Network Effects and Dynamics of Competition

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PRELIMINARY and INCOMPLETE

Comments are wellcome!

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Abstract

I analyze the dynamic price competition in a horizontally differentiated duopoly in the presence of network effects with two incompatible durable products. Consumers value current and future installed bases. I derive sufficient conditions for the existence of stable Markov-Perfect Equilibrium(MPE) in affine strategies when the market is covered and shared each period. When they exist, the optimal pricing policies suggest that a firm with a higher previous market share charges a higher price, all else equal. It is possible to observe pricing below cost for some periods. In the long-run, the MPE leads to a more competitive outcome (lower average prices) than the case where there are no network effects. Imposing compatibility may reduce consumer surplus.

Keywords: Dynamic price competition, network externalities, fashion, markov-perfect equilibrium, product differentiation.

JEL Classification: C73, D21, D43, L13, L21.

1 Introduction

For some products the value of consumption may increase with the size of the user base. The most immediate example is communication platforms. Due to technological standards, people might consider the installed base of a particular technology as well as its future popularity when they make purchasing decisions. If communicating with others is a necessity, the platform which hosts a large number of users, presents a larger number of potential communication partners. Thus, more users a particular communication platform has the more valuable it will become for current and future users.

Similar effects could be observed in any market where some products deliver value in combination with others. Computer operating systems present a timely and fashionable example of such markets given the recent Microsoft case. An operating system is valuable only in combination with applications software. Vendors of applications software will naturally develop their products for operating systems which have a large market potential, that is a large current and expected user base. It could then be deduced that a computing platform which is and expected to be popular will have a larger variety of application software. A larger variety of software in turn will induce more people to consider adopting the platform in consideration closing the feedback loop. Therefore, an operating system which has a large market potential will be more valuable.

These feedback effects are termed *network effects* in the literature. In this paper, I will investigate the dynamics of markets for such products. I develop a highly stylized model of a differentiated products duopoly which exhibit network effects for two incompatible durable products. There are a few differences of the model presented below to those found in the literature. One of my goals is to develop a framework and derive conditions where two incompatible products are sold side by side in the market. The literature offer several dynamic multiperiod models such Farrell and Saloner (1986) and Katz and Shapiro (1995) where the demand is served only by one firm. An important difference of this paper is that the expectation about future size of firms' networks are formed rationally, that is the current prices **do** effect the beliefs of consumers about future popularity of a product. Moreover, I adopt a heterogeneous consumer population with random tastes which allows for both products to co-exist, as well as introduce a mechanism where random shocks effect the evolution of market structure.

The model in consideration consists of two symmetric firms which produce horizontally differentiated, durable and incompatible products with constant marginal cost. A product differentiation model á la Hotelling is adopted; in addition, the consumers' valuation includes a term that reflects the effects of the previous, current and future market shares of a particular brand. There are two overlapping generations of consumers of which only the new ones make a purchase when they arrive to the market. Once a brand is bought it delivers utility for the lifetime of consumers and switching to other brands are assumed to be impossible due to high switching costs. They observe current installed bases and prices, then form rational expectations for current and future allocation of consumers between brands. Firms compete for these consumers in prices taking into account the intertemporal effects of their pricing policies. A firm with a large installed base faces a tradeoff between exploiting high valuation it receives from the consumer population today and foregoing future installed base advantage due to high prices this period.

The stylized model analyzed in the paper fits in to the category of linear-quadratic dynamic games, hence, I consider affine Markov strategies for both firms. The analysis is restricted to the case where market is covered and shared in each period. I establish sufficient conditions required for the existence of a Markov Perfect Equilibrium(MPE) in affine strategies.

In equilibrium, a firm with a higher installed base charges a higher price, all else being equal. In the long-run, average prices are lower than what they would have been in the absence of network effects and/or when products are compatible. Moreover, it is optimal to price below cost for certain parameter values and very small installed bases. The welfare effects of imposing compatibility is ambiguous, however there is range of parameters—relatively strong network effects—where imposing compatibility reduces welfare. This is due to the increase in prices with full compatibility; the gains in terms of network benefits is less than the increase in prices.

The paper is organized as follows. The related literature is presented in section 2. The model is outlined in Section 3. The analysis of the model is presented in Section 4. Section 5 concludes.

2 Related Literature

In this section my goal is provide a brief review of the literature on network effects. For a more detailed account of the literature on network effects and compatibility decisions on market outcomes Katz and Shapiro(1994) and Economides (1996) are two excellent surveys .

A very early paper dealing with effects of past sales on current market outcomes is von Weiszäcker (1971). He concludes that if indeed the consumer preferences evolve endogenously, policy decisions are better based on steady state preferences and derives conditions on the preferences so that a steady state is achieved. Another early analysis of network effects on consumer demand for telecommunications can be found in Rohlfs (1974) where he analyzed the demand for subscribtions to the telephone networks.

Nevertheless, the seminal treatment of markets with network effects is Katz and Shapiro (1985) which documents many of the intrinsic problems with the help of a simple and clear model. Consumers value the expected size of the network of each product. They show that for each set of expectations, there is a corresponding equilibrium. The question of how expectations form has many reasonable answers. Katz and Shapiro (1985) refine the analysis by imposing a condition that expectations must be fulfilled in equilibrium. In their model, the expectations are formed before firms make strategical decisions and are unaffected by them.

The fulfilled expectations equilibrium prices under different compatibility structures are then derived. For a given number of firms the prices increase in the expected network size and as the number of firms increase, prices approach marginal cost levels. Full compatibility implies a unique equilibrium, while in case of incompatibility, the predictions of their model are multiple. There may be a symmetric market equilibrium where all firms are active; a natural oligopoly might arise where some of the firms exit; and, more surprisingly, there may be asymmetric outcomes. Asymmetric equilibria are particularly interesting, as the only differences between the firms are the differences in the consumers' ex–ante expectations.

Moreover, they find that when costs of achieving complete compatibility are fixed, any move towards compatibility that enhances industry–wide profits is socially desirable. However, it is not always the case that firms have incentives to achieve full compatibility. Firms may fail to move towards compatibility, even when it is socially desirable.

Farell and Saloner (1986) presents a dynamic model analyzing the adoption of technologies which exhibit network effects in a continuous time framework. They concentrate the consumer side of the problem by assuming perfectly competitive provision of the network good for most of the paper and show that even a superior technology may not be adopted.

Katz and Shapiro (1986) investigate the effects of network externalities on the adoption of new technologies in a two period model. The main results are that in the presence of network externalities history matters and an inferior technology, which has a higher installed base, may survive an entrant with a superior technology. Farrell and Saloner (1992) analyze the role of converters on market outcomes as well as the incentive of firms to provide them.

Katz and Shapiro (1992) studies the problem of product introduction with network externalities in a homogeneous goods continuous-time dynamic duopoly model. In equilibrium, the market is always served by one of the firms for the remainder of the time. Which firm gets the market forever, critically depends on the relative costs advantage of the new technology, installed base of the incumbent as well as the strength of the consumer confidence. The model assumes that once consumers buy a brand, they face infinite switching costs, therefore they are locked in to a single brand.

An important feature of models of consumer behavior with network externalities is that, consumers have to form expectations about the future network sizes. Katz and Shapiro (1985) deals with fulfilled expectations which is to say that prices do not effect expectations. In both Farrell and Saloner (1986) and Katz and Shapiro (1992) the expectations are extreme in the sense that all current and future consumers agree on one firm, thus market is served at all times by one firm or one technology. The prices do not have any effect on consumer expectations. In this paper, I assume that expectations are formed after price information is received thus prices **do** effect expected demands. Moreover, even though market size remains constant over time, consumers have random tastes, thus what is expected and what is realized differ. However, given information on prices and installed bases firms and consumer agree on expectations about current and future demand. Moreover, the assumed consumer heterogeneity implies that there may more than one technology or firm serving the market. For this to occur however, as will be shown below, network effects must be sufficiently small.

3 The Consumer Choice

The model involves two firms producing horizontally differentiated durable products. At each period, there are two overlapping generations formed of K customers each with a life span of two periods. They are labelled young and old.

Each consumer derives a stand alone utility of R for consuming one unit of one of the brands, i.e. both products have the same stand alone value.[1] The consumption takes place as soon as they arrive to the market—assuming a high enough R would guarantee immediate purchase. Durability of the products imply, once bought the products can be used for two periods, that is for the whole lifetime of a consumer. In any case, even if a consumer is not too happy with what they have purchased, the switching costs are assumed high enough that no one would switch to the other brand when they become old. Thus, old consumers have no decision to make. They keep using the product they already have and derive the corresponding benefits.

Each member of the young generation arrive with an affinity towards one of the brands. This affinity is modelled á la Hotelling. It is assumed that the product of firm 1 is located at zero, while the second firm is located at one or formally $L_i = i - 1$. Every customer draws a location in the unit interval from a uniform distribution, and their distance to each firm implies the value discrepancy between their ideal product and the available products. Thus, they incur a transportation cost which represent their affinity; at equal prices, each consumer prefers the brand she is closer to. The draws of consumers are independent from the draws of others. This affinity could be due to differences in observed marketing, different social circles, uncertainty about R and so on. Thus, the transportation cost is only due to the perception of the products and play a role only at the moment of purchase, and is irrelevant after purchase. The per unit travel cost is denoted by c.[2]

In addition to this standard horizontal differentiation model, the customers have a network benefit component, $N(\cdot)$, in their valuation. The more users belong to the customer

[1] The product innovations are assumed away in the paper, however one could think of an exogenous innovation process adopted by both firms simultaneously and consumers update the stand alone utility accordingly.

[2] Observe that c here provides a measure of substitutability between both brands. When $c \to 0$, the transportation cost approaches to zero, therefore both brands become close substitutes. On the other hand for $c \to \infty$, the transportation cost approaches infinity, therefore consumers prefer the closest brand independent of their prices.

base of a product the more valuable it is. The products are assumed to be incompatible, thus each firm's own customer base is considered when valuing its product. This effect is assumed to be a linear function of the market share of each product. Clearly, each firm's market share will be composed of some consumers belonging to the old generation and others to the young. Formally, the network benefit is $N(m_i^t) = \bar{n}m_i^t$, where m_i^t denotes the market share of firm i in period t. Here, $\bar{n}$ is a measure of the strength of network externalities.

Young consumers, on the other hand, have to make purchasing decisions taking into account the benefits they will receive when they are old. The benefits in the second period of their lives are discounted with a rate δ. They will have to form expectations about the behavior of their own cohort as well the next cohort. Thus, it is useful to decompose the market share of each firm in to a share of young consumers, y_i^t, and old o_i^t. Clearly, $o_i^t = y_i^{t-1}$, thus it is sufficient to consider market shares of young consumers in subsequent periods. A consumer located at α derives a net utility given by

$$\begin{aligned} U(\alpha, \bar{p}_i^t, y_i^{t-1}, y_i^{t,e}, y_i^{t+1,e}) &= R + \bar{n}\big(y_i^{t-1} + y_i^{t,e}\big) - \mid \alpha - L_i \mid c - \bar{p}_i^t \\ &\quad + \delta \left[R + \bar{n}\big(y_i^{t,e} + y_i^{t+1,e}\big) - \mid \alpha - L_i \mid c \right], \end{aligned} \tag{1}$$

where $y_i^{t,e}$ refers to the expected share of cohort t buying firm i's product.

Every period each young consumer buys one unit of the product which provides the highest value. That is, the customer located at α chooses product i over j if and only if

$$U(\alpha, \bar{p}_i^t, y_i^{t-1}, y_i^{t,e}, y_i^{t+1,e}) > U(\alpha, \bar{p}_j^t, y_j^{t-1}, y_j^{t,e}, y_j^{t+1,e}).$$

One crucial assumption I will make is that, the market is covered and shared each period. That is both firms have positive sales which add up to one. Therefore, $y_j^t = 1 - y_i^t$, and it is sufficient consider only the market share of firm 1 to describe the market configuration. From this point on I will use y^t to denote the share of young consumers who prefer firm 1 at time t.

How the expectations are formed is a very difficult question with many answers. In the literature, several concepts have been extensively used: rational expectations and fulfilled expectation. In this context, the difference amounts to the role played by prices in the formation of expectations. I will assume that consumers are able to form rational expectations which are formed after observing installed base and prices of both firms. Therefore, firms could affect the expectations by their strategical pricing decisions. Rationality of expectations implies that consumers can calculate the market shares generated by their own cohort as well as they could correctly infer future prices enabling them to correctly infer the expected future market shares.

Following the standard procedure, I look for the location of the indifferent customer, whose location is denoted by $\tilde{\alpha}$, to find the demand functions for each product. Observe that all the customers to the left of $\tilde{\alpha}$ prefer product 1 while customers to the right prefer product 2. Thus, I will start with the assumption that rational expectations imply $y^{t,e} = \tilde{\alpha}$, that is exactly a fraction $\tilde{\alpha}$ of the K customers are rationally expected to join network 1.

The young generation each period does not know the prices which will prevail next period, however they could correctly infer them in equilibrium. These prices will critically affect how the young will decide next period. I will assume, for now, a rule which allows customers today to form expectations about the network sizes due to consumption of the young in the next period. This rule is based on the existing stock of old consumers for each firm, that is $y^{t+1,e} = f(y^t)$, furthermore I assume this function to be affine in its argument. Formally

$$f(y^t) = a + by^t,$$

and to ensure that this form is consistent with the covered and shared market assumption it must be that $f(1 - y^t) = 1 - f(y^t)$. This requirement imposes a restriction on the parameters of the function $f(\cdot)$. Solving for this restriction yields to $b = 1 - 2a$. If firms are aware of this rule for forming expectations, choose their prices accordingly, and in return, consumers make their network choices in line with the rule, the expectation will be rational in equilibrium. Thus, the parameter a becomes an integral part of the market equilibrium. Using this rule, one can write

$$y^{t+1,e} = a + (1-2a)y^{t,e} = a + (1-2a)\tilde{\alpha}.$$

Substituting for both $y^{t,e}$ and $y^{t+1,e}$ in (1) for both products, and solving for $\tilde{\alpha}$ leads to the location of the indifferent consumer which is given by

$$\bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1}) = \frac{1}{2} + \frac{1}{2}\frac{\bar{n}(2y^{t-1}) + \bar{p}_2^t - \bar{p}_1^t}{c - (1+2\delta)\bar{n} + 2\delta\bar{n}a}. \tag{2}$$

Given the uniform distribution of consumer tastes (i.e., affinities), prices and the stock of old consumers, each consumer will subscribe to network 1 with probability $\bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1})$ and to network 2 with probability $1 - \bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1})$. The market share of firm 1 can take one of the $K+1$ values with a certain probability, thus it is a random variable. Let us denote this random variable with $y^t = \tilde{m}/K$ where $\tilde{m}$ is the realization of the number of new subscribers network 1 has. y^t will take this value only when m out K customers draw taste variables that are less then the location of the indifferent consumer, that is $\alpha_i \leq \tilde{\alpha}$ for $i \in I$ and $|I| = m$, and $\alpha_j > \tilde{\alpha}$ for $j \in J$ and $|J| = K - m$, with $m \in \{0, 1, \dots, K\}$. There are K choose m different ways to allocate consumers between the firms, but given a particular allocation the probability of drawing appropriate taste variables is the same. Then, the probability that y^t being equal to m/K is given by

$$\begin{aligned} P(y^t = \frac{m}{K}) &= \frac{K!}{m!(K-m)!}\Pi_{i\in I}P(\alpha_i \leq \tilde{\alpha})\Pi_{j\in J}P(\alpha_i > \tilde{\alpha}) \\ &= \frac{K!}{m!(K-m)!}\bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1})^m(1 - \bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1}))^{K-m} \end{aligned} \tag{3}$$

Given the probabilities in (3), it is straightforward to compute the expected value of y^t which amounts to

$$E(y^t) = \bar{d}(\bar{p}_1^t, \bar{p}_2^t, y^{t-1}). \tag{4}$$

This property justifies the use of the location of the indifferent consumers as the rational expectation of the market share of network 1 in calculating (2).

It is useful to realize at this point that when prices and the magnitude of network effects are measured in units of c, the demand remains unchanged.[3] Hence, from this point on n will denote the ratio of the magnitude of network effects and the level of transportation costs, i.e $n = \bar{n}/c$, and transportation costs are assumed to be one. In addition, defining $p_i^t = \bar{p}_i^t/c$, the expected market share of network 1 can now be written as

$$d_1(p_1^t, p_2^t, y^{t-1}) = \frac{1}{2} + \frac{1}{2}\frac{n(2y^{t-1}) + p_2^t - p_1^t}{1 - (1+2\delta)n + 2\delta na}. \tag{5}$$

with the redefined units. This allows me to simplify the exposition of the algebra considerably. Then, the demand for the product of firm 1 is simply $D_1(p_1^t, p_2^t, y^{t-1}) = d_1(p_1^t, p_2^t, y^{t-1})K$. For this function to be downward sloping, it is necessary that

$$1 - (1+2\delta)n + 2\delta na > 0, \tag{6}$$

or, equivalently,

$$1 - \frac{1-n}{2\delta n} < a. \tag{7}$$

Observe that a is a parameter which will be known in equilibrium, therefore once the candidate equilibrium strategies and the corresponding value of a are found, it must be checked whether (7) holds. This will impose further restrictions on the model parameters for an equilibrium of the type where market is covered and shared each period.

Clearly, $y^{t,e} = E(y^t) = d_1(p_1^t, p_2^t, y^{t-1})$, and determines expected value of the fraction of consumers which will form the customer base of firm 1 next period. Moreover, given that the market is covered and shared each period, the expected market share for the product of network 2 is given by

$$d_2(p_1^t, p_2^t, y^{t-1}) = 1 - \bar{d}_1(p_1^t, p_2^t, y^{t-1}). \tag{8}$$

And the expected demand is $D_2(p_1^t, p_2^t, y^{t-1}) = (1 - d_1(p_1^t, p_2^t, y^{t-1}))K$. Therefore, the

[3]This is just a normalization of utilities, and hence the underlying preferences remain unchanged leading to the same demand functions.

market share of firm 1 completely characterizes the market structure, thus I drop the subscript and denote the market share of firm 1 by $d(p_1^t, p_2^t, y^{t-1})$

4 The firms

For simplicity, I assume firms incur zero marginal costs, however, this assumption can be relaxed given the linearity of utilities.[4] Per period profit functions[5] of the firms are given by

$$\pi_1(p_1^t, p_2^t, y^{t-1}) = p_1^t D_1(p_1^t, p_2^t, y^{t-1}) \tag{9}$$

$$\pi_2(p_1^t, p_2^t, y^{t-1}) = p_2^t D_2(p_1^t, p_2^t, y^{t-1}) \tag{10}$$

Observe that while y^{t-1} is observed, the demand from the current generation can only be known in expectation at the time prices are set, thus the above per period profit functions are actually expected values of the per period profits.

The competition has infinite horizon. For simplicity, I assume that firms have a common discount factor which is the same as the discount factor used by consumers, δ. Each firm chooses a price path to maximize the discounted sum of its profit streams. The objective function of each firm is given by

$$\Pi_i^t(y^{t-1}) = E_0\left[\sum_{j=0}^{\infty} \delta^j \pi_i(p_1^{t+j}, p_2^{t+j}, y^{t+j-1})\right], \qquad i = 1, 2. \tag{11}$$

Notice that, already at time $t = 0$, firms has to use an expected objective.

5 The Analysis

The interaction between the firms constitute a dynamic game. That is, the demand faced by each firm is a function of their customer base,y^{t-1}, therefore the stage game they face

[4]In fact, when the marginal costs of both firms are the same, the model will provide the same results if one views the strategic variable p_j^t as the markup above cost instead of prices.

[5]Recall that prices are measured in units of transportation cost c.

changes according to the changes in their customer bases. Customer base of firm 1 at time t,y^{t-1} is a natural state variable. Therefore, I will look for stationary pricing policies as functions of the customer bases for each firm, i.e. for stationary markovian strategies.

I also require that such policies form a subgame perfect equilibrium. Hence the equilibrium concept adopted here is Markov Perfect Equilibrium (MPE). (cf. Maskin and Tirole (1987), (2000) as well as Basar and Oldser (1982)) In a MPE, given the policy of the opponent, each player solves a dynamic programming problem. Equilibrium is established whenever policies of each player forms best responses to each other.

One can find closed form expressions for equilibrium policies only under very specific functional forms. One such form is obtained, whenever

- Per period payoffs are quadratic in the strategic variables and state as well as concave in own strategies,
- The state evolves as a linear function of strategies and state.

Hence the well-known term Linear Quadratic(LQ) games is used to refer to such games. These games possess several important and well known properties:

- Strategies of both players are affine functions of the state
- Given these strategies, the discounted sum of profits, the Value Functions, are quadratic in the state variable.

There are few important points to note. First, one crucial assumption underlying the model is that market is shared and covered at every period. This assumption requires to account for constraints imposed on the demands each period. Dealing with dynamic games with constraints on the state variable is very complicated even for LQ games.

Second, firms prefer producing every period. Since negative prices are allowed, it is not trivial that they will earn non-negative profits. Third, the expectations formed consumers have to lead to an economically sensible demand function, that is, a has to take a value such that (7) holds in equilibrium.

The strategy I will follow to derive equilibrium pricing policies and expectations is to ignore these constraints when solving the coupled dynamic programming problems at first. After characterizing the policy functions, I will derive conditions on the model parameters such that these constraints will never become active when firms follow their equilibrium policies and consumers form their expectations using $f(\cdot)$.

Thus, the type of equilibrium I am trying to establish does not exist for all parameter values. In fact for some parameter values such as very high magnitude of network effects, one firm will be able to corner the market whatever the initial state is. This equilibrium resembles to that found in Katz and Shapiro (1992).

It is useful to briefly discuss the incentives of a firm when choosing its price in each period. Inspection of (5) and (8) reveals that a firm with a market share larger than a half faces a higher demand function due to the valuation of previous market shares by consumers. Therefore, a firm might be able to sustain a high price at a given period, but choosing a higher price may decrease market share in that period leading to a lower demand in the future. Hence, in equilibrium the contrasting incentives of higher profits this period versus lower demand in the future must be balanced.

As customary with LQ games, I will assume firms choose symmetric policy functions[6] of the form

$$\begin{aligned} p_1(y) &= l + ky, \\ p_2(y) &= l + k(1-y). \end{aligned}$$

Substituting these pricing policies in (5) will reveal that indeed the assumption on the

[6]One can start with asymmetric policy functions and indeed show that the resulting equilibrium will be symmetric. It is also possible start with an assumption of symmetry and as long as the result confirms this assumption, the solution methodology remains valid.

formation of consumer expectations will be justified whenever

$$k = \delta\ (2\,a - 1)^2\, n + (1 - \delta\, n - n)\,(2\,a - 1) + n.$$

In this case, the value functions will take the form

$$\begin{aligned} V_1(y) &= r_0 + r_1 y + r_2 y^2, && (12)\\ V_2(y) &= r_0 + r_1(1-y) + r_2(1-y)^2. && (13) \end{aligned}$$

The parameters of the the policy functions, (l, k), and the value functions, (r_0, r_1, r_2), and the expectations, a, are to be determined to characterize the equilibrium.

Firm i, $i = 1, 2$, solves the following problem

$$\left\{ V_1(y_i^{t-1}) = \right\} \max_{p_i} \left[\pi_i(p_i, p_1(y_i^{t-1}), y_i^{t-1}) + \delta E[V_1(y_i^t)] \right], \qquad (14)$$

with

$$y_i^t = \bar{d}(p_i, p_j(y^{t-1}), y_i^{t-1}),$$

with $j \neq i$. The expectation of the value function has to be computed. Notice that y^t, the realized value, is a discrete random variable which takes $K+1$ values each with the corresponding probability $P(y^t = m/K)$, $m = 0, \ldots, K$. It is sufficient to find the first and second moments of y^t given the postulated quadratic value functions. We know the first moment from (4), hence we need additionally the second moment—$E([y^t]^2)$. A simple calculation leads to

$$E([y^t]^2) = \frac{1}{K}\tilde{\alpha} + (1 - \frac{1}{K})\tilde{\alpha}^2.$$

Let us denote with $EV(\cdot)$, the expected value of the value function, then the counterparts of (12) and (13) become

$$\begin{aligned} EV_1(y) &= r_0 + (r_1 + \frac{r_2}{K})y + r_2(1 - \frac{1}{K})y^2, && (15)\\ EV_2(y) &= r_0 + (r_1 + \frac{r_2}{K})(1-y) + r_2(1 - \frac{1}{K})(1-y)^2. && (16) \end{aligned}$$

For these maximization problems stated in (14), first order conditions(FOCs) are necessary and sufficient. Whenever, $p_1(y^{t-1})$ and $p_2(y^{t-1})$ turn out to be best responses to each other, the value functions will satisfy the equalities given in curly brackets, with the right hand sides evaluated at the optimal policies. This feature makes it possible to calculate the parameters of the value functions as functions of the parameters of the model primitives using the method of undetermined coefficients.

Lemma 1. *The policy function parameters which satisfies the FOCs are given by*

$$k^*(a) = \delta\ (2\,a-1)^2\,n + (1-\delta\,n-n)\,(2\,a-1) + n \tag{17}$$

and

$$l^*(a) = -\frac{1}{2}\,(1-\delta)\,k^*(a) + \frac{k^*(a)-n}{+2\,a-1} - \delta n, \tag{18}$$

and the parameters of the value functions consistent with $(l^*, k^*(a))$ *are given by*

$$r_2^*(a) = \frac{K^2\,(3\,k^*(a)-2\,n)}{2\,(K-1)\,\delta\ (2\,a-1)} \tag{19}$$

$$r_1^*(a) = -K\ (k^*(a)-2\,n)\,a, \tag{20}$$

$$r_0^*(a) \quad = \quad \frac{a^2 K\ (k^*(a)-2\,n)}{2\,(2\,a-1)\,(1-\delta)}, \tag{21}$$

where a *solves the following equation:*

$$g(a) = \frac{K^2\,(3\,k^*(a)-2\,n)}{2\,(K-1)\,\delta\ (2\,a-1)} - \frac{1}{2}\,K\ (2\,a-1)\,(k^*(a)-2\,n) = 0. \tag{22}$$

Proof. See appendix.

If these policy functions indeed constitute andequilibrium then the state, the expected stock of young customers of firm 1, evolves according to a difference equation given by

$$E[y^t(y^{t-1})] = a + (1-2a)y^{t-1},$$

and hence, the consumer expectations are rational. However, for every value of $y^{t-1} \in [0,1]$, it is required $E[y^t(y^{t-1})] \in (0,1)$ in order to satisfy the assumption on market being covered and shared each period. It is easy to verify that as long as $\mid (1-2a) \mid < 1$, the expected value of the next state always falls in the open interval $(0,1)$. Presented differently, the market will be covered and shared whenever $0 < a < 1$. Recall also that for demand function to be sensible, it is necessary that (7) holds. Observe that the left hand side of (7) is negative whenever

$$n < \frac{1}{1+2\delta} = n_0(\delta). \tag{23}$$

Therefore, provided that (23)holds, the expectation parameter a should be solution of $g(a) = 0$ such that $a \in (0,1)$. It is easy to show that $g(0) \geq 0$ for $n \leq n_1(\delta, K)$ where

$$n_1(\delta, K) = \frac{1}{2}\frac{3\,K - \delta\,(K-1)}{2\,K + \delta\,(3\,K - \delta\,(K-1))}.$$

Note that it is easy to show that $n_1(\delta, K) < n_0(\delta)$ for all $K > 1$.

Lemma 2. (Roots of $g(a)$) *There exists a unique root of $g(a)$ in $(0, 1/2)$ and no root of $g(a)$ in $[1/2, 1]$, whenever*

$$n < n_1(\delta, K) < n_0(\delta) < 1.$$

Proof. See appendix.

There are values of n where $g(a)$ has two roots in $[0,1)$, particularly when $n = n_1(\delta, K)$, then $a = 0$ solves $g(a) = 0$. Even though interesting, the region where there are two roots are relatively small compared to the region described in Lemma (2), and the qualitative properties of the equilibrium are similar, thus I do not analyze these cases in detail.

Lemma 3. *When there is a unique root of $g(a)$ in $(0,1)$, $0 < k^*(a^*) < \frac{2n}{3}$. This implies that $r_2(a^*) > 0$, $r_1(a^*) > 0$ and $r_0(a^*) > 0$.*

Proof. See appendix.

The positive values of the coefficients of the value functions guarantee that in the candidate equilibrium, both networks have positive expected profits. Moreover, expected profits are increasing in the installed base. Also notice that, whenever $n = n_1(\delta, K)$, $a = 0$ is a root of $g(a)$. It is easy to see that $l^*(0) < 0$. Thus, in a neighborhood of $n_1(\delta, K)$, a firm with very small market share may charge negative, i.e. below cost, prices.

Proposition 1. (Existence of Markov Perfect Equilibrium) *Assume that the conditions of Lemma 2 is satisfied. Then, there exists a symmetric MPE in stationary affine strategies where market is covered and shared each period. The parameters of the affine policy functions are given by Lemma 1. In this equilibrium,*

i. The firm with a higher customer base charges a higher price, i.e $k^*(a^*) > 0$.

ii. The long-run average of market shares are one half, i.e. the after sufficiently many periods market shares oscillate around one half.

iii. There is nonempty set of parameters and stock of old consumers where pricing below cost is the optimal policy.

Proof. See appendix.

As in finite period models, for example Farrell and Saloner (1992),, there is a range of parameters where the network effects are not too strong for which a unique MPE exists. It is possible to give examples of markets where two or more propriety technologies existing side by side for longer periods of time. An immediate candidate is the market for video games. Some application software markets also can be seen as examples of markets where incompatible products which exhibit network externalities—albeit small—are sold side by side. The market for streaming media players could also be considered an example where two products, Windows Media Player and RealOne player, share the market. It is naturally interesting to look at the rest of the parameter space, however this is significantly more complicated due to potential discontinuties.

One important character of the model is that there is no steady state. That is, there is a randomness in the market outcome every period, due to consumers randomly determined tastes/affinities. A particularly biased realization could change the market position of each network every period. However, both firms are correct in their assessments of market conditions in expectations.

The network with a larger user base is valued more by the consumers, thus it is able to charge a larger price. This is similar to other models of competition with network effects. A point which is not immediately clear is the comparison of prices with those of a market without network effects. There will be no state dependence in the consumer demand in the absence of network effects, thus the only MPE is the repetition of the equilibrium of the stage game, in which both firms will charge a price equal to the transportation cost which is normalized to one in the model presented above, i.e. $p_{n=0} = 1$. Moreover, with full compatibility expected network benefits always amount to one, thus full compatibility equilibrium will be exactly as the equilibrium with no network benefits.

As mentioned before, in this model there is no steady state due randomness in the formation consumer affinities. Nevertheless, as stated in part *ii* of the Proposition 1, after a sufficiently long period of time the time-average of market share of each firm will approach one half. This is also true when network effects approach zero or with full compatibility. Thus, it might make sense to compare prices when each firm has a customer base of one half. Formally, the prices charged in this case are

$$\bar{p} = p_1(1/2) = p_2(1/2) = l^*(a) + \frac{1}{2}k^*(a) = \frac{\delta\, k}{2} + \delta\, n\,(2\,a - 1) + 1 - 2\,\delta\, n - n, \tag{24}$$

while when $n = 0$

$$p_{n=0} = 1.$$

Proposition 2. (Long-Run Average of Prices)

The market shares in the long run oscillate around one half. The average prices, $\bar{p}$, satisfy

$$\max(1 + \delta^2 n - \frac{1}{2}\,\delta - 2\,\delta\, n - n) \leq \bar{p} \leq 1 - n - \frac{3}{2}\,\delta\, n \leq p_{n=0} = 1,$$

whenever $n < n_1(\delta, K)$. Thus, in the presence of network effects long term average of prices are less than what they would have been in the absence of network effects.

Proof. See appendix.

Proposition 2implies that dynamic competition in the presence of network effects is rather fierce, in fact fiercer than a market without network effects or a market with network effects but full compatibility. The fact that network effects increases competitiveness in a market is rather surprising but based on economically sound reasons. An investigation of (5) reveals that each firms demand shifts outwards with customer base they carry over to the next stage. Also observe that $1 - (1 + 2\delta)n + 2\delta na < 1$ for all $a \in [0, \frac{1}{2}]$ and $0 \leq n < n_1(\delta, K)$, thus the demand is more price sensitive. Therefore, a small price cut could increase the customer base next period well above one half. This incentive is present for both firms and even stronger in the long run since they would have market shares close to one half. When these incentives are balanced in equilibrium, the prices turn out to be lower than a market where these incentives are absent.

The transition to long run behavior is rather smooth for most parts of the parameter region. There is, however, a nonempty set of parameters where the equilibrium value of the parameter of consumer expectations, a^*, is very close to zero. The implication of a very low value of a^* is that expected market shares are going to change very slowly. In fact, in the limiting case of $a^* = 0$ the rational expectations prescribe that *"whatever happened in the previous period will happen this period"*.

For such parameters, one biased realization of consumer tastes may change the market structure for a long while. That is, dominance due to a biased realization of of consumer tastes may be sustained for a long while. Moreover, the dominant firm will be able charge higher prices than its rival. Hence, for an outside observer, it might seem that the market is dominated by a firm who can sustain high prices due to anti-competitive behavior and thus hurtful for consumers. This, however, would be an utterly mistaken conclusion, as it is this case where the long run average prices come very close to marginal cost levels.

The socially optimal allocation of consumers would imply symmetric market shares and marginal cost prices. Thus, there would be some welfare loss due to asymmetric market shares. It is not clear however whether it would be socially desirable enforce compatibility in such a market. This would allocate consumers symmetrically between networks, increase network benefits, but at the same time increases average prices.

In the present model, two compatible networks will share the market equally and and charge prices of one which is the level of the Hotelling transportation cost. Evaluating a long run average per period consumer surplus in this case amounts to

$$CS^{Compatible} = (1+\delta)(R+2n) - 1 - \frac{1}{4}$$

where the last term, 1/4, represents the aggregate transport incurred. The average per period welfare in the model presented above, on the other hand, is

$$CS^{Incompatible} = (1+\delta)(R+n) - \bar{p} - \frac{1}{4}.$$

Thus, for compatibility to be preferable it is necessary that

$$CS^{Compatible} - CS^{Incompatible} = n - (1-\bar{p}) > 0.$$

That is, the network benefits due to compatibility must outweigh the change in prices.

Proposition 3. (Compatibility may reduce welfare.)
For $n < n_1(\delta, K)$ and $n \to n_1(\delta, K)$, imposing compatibility may reduce consumer surplus.

Proof. As stated in Proposition 2 $\bar{p} < 1 - n - \frac{3}{2}\delta n$, thus when prices are closer to the upper bound, then compatibility is preferable. However, if prices are closer to lower bound, which would be the case when a^* is close to zero, $n - (1 - \bar{p}_{min}) = -1/2\,\delta\,(1 + 4\,n - 2\,\delta\,n) < 0$. Therefore, imposing compatibility will reduce average consumer surplus due to its effect on prices. The effects on profits are more difficult to compare. ■.

The reduction in consumer welfare occurs when network effects are relatively strong. Unfortunately, this is exactly the case where intuition would suggest that compatibility is preferred by customers.

insert simulations

6 Conclusion

I have analyzed a stylized model of competition in durable goods-market with network effects. For more than one technology to be present in the market, it is necessary that the network effects are not too large relative to the heterogeneity in the consumer tastes. I derived conditions for the existence of a MPE. In this equilibrium a network with a larger user base quotes a larger price. It is possible to observe below cost pricing for some range of parameters.

The randomness of consumer tastes leads to interesting patterns of evolution of market shares and prices. It is possible that expectations change very slowly, which in turn leads to dominance of one of the firms after a rather asymmetric shock to consumer tastes. The market shares oscillate around one half in the long run, but due to randomness there is never a steady state.

The most surprising result is the long run average of prices converge to a level below what it would have been in the absence of network effects or full compatibility. Thus, network effects for incompatible products make markets very competitive. This raises interesting policy questions regarding standardization. Imposition of a standard may reduce consumer surplus contrary to conventional wisdom. In such cases network effects are sufficiently high to increase incentives of firms to reduce prices under incompatibility, but small enough that the market is shared in equilibrium. Standardization in this case softens competition, leading to an increase in prices, however the gains of standardization due to increased network benefits is not sufficient to account for this price hike.

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Appendix

Lemma 1. *The policy function parameters which satisfies the FOCs are given by*

$$k^*(a) = \delta\ (2\,a-1)^2\,n + (1-\delta\,n-n)\,(2\,a-1) + n \tag{17}$$

and

$$l^*(a) = -\frac{1}{2}\,(1-\delta\,)k^*(a) + \frac{k^*(a)-n}{2\,a-1} - \delta n, \tag{18}$$

and the parameters of the value functions consistent with $(l^*, k^*(a))$ *are given by*

$$r_2^*(a) = \frac{K^2\,(3\,k^*(a)-2\,n)}{2\,(K-1)\,\delta\ (2\,a-1)} \tag{19}$$

$$r_1^*(a) = -K\ (k^*(a)-2\,n)\,a, \tag{20}$$

$$r_0^*(a) \quad = \quad \frac{a^2K\ (k^*(a)-2\,n)}{2\,(2\,a-1)\,(1-\delta)}, \tag{21}$$

where a solves the following equation:

$$g(a) = \frac{K^2\,(3\,k^*(a)-2\,n)}{2\,(K-1)\,\delta\ (2\,a-1)} - \frac{1}{2}\,K\ (2\,a-1)\,(k^*(a)-2\,n) = 0. \tag{22}$$

Proof. Firm i, $i = 1, 2$, solves the following problem

$$\left\{V_i(y^{t-1}) = \right\} \max_{p_i} \Big[\pi_i(p_i, p_j(y^{t-1}), y^{t-1}) + \delta E[V_i(y^t)]\Big], \tag{14}$$

with

$$E[y^t] = d(p_i, p_j(y^{t-1}), y^{t-1}),$$

with $j \neq i$ and

$$E[V_i(y_i)] \;=\; r_0 + (r_1 + \frac{r_2}{K})y_i + r_2(1 - \frac{1}{K})y_i^2, \tag{15}$$

$$\tag{16}$$

Solving both optimization problems, and also solving (l, k) in terms of (r_1, r_2), it is easy to verify that the equilibrium prices will be

$$\begin{aligned} p^* &= \frac{n\left(-2\,K^2\delta\,na - K^2 + 2\,K^2\delta\,n + \delta\,r_2\,K + K^2n - \delta\,r_2\right)}{-6\,K^2\delta\,na + 6\,K^2\delta\,n + 3\,K^2n - 2\,\delta\,r_2 - 3\,K^2 + 2\,\delta\,r_2\,K}\left(-1 + 2\,y^{t-1}\right) \\ &\quad - \frac{\delta\,r_2 - 2\,K\,\delta\,na + 2\,K\,\delta\,n + \delta\,r_1 - K + K\,n}{K} \end{aligned} \tag{25}$$

$$\begin{aligned} q^* &= \frac{n\left(-2\,K^2\delta\,na - K^2 + 2\,K^2\delta\,n + \delta\,r_2\,K + K^2n - \delta\,r_2\right)}{-6\,K^2\delta\,na + 6\,K^2\delta\,n + 3\,K^2n - 2\,\delta\,r_2 - 3\,K^2 + 2\,\delta\,r_2\,K}\left(1 - 2\,y^{t-1}\right) \\ &\quad - \frac{\delta\,r_2 - 2\,K\,\delta\,na + 2\,K\,\delta\,n + \delta\,r_1 - K + K\,n}{K} \end{aligned} \tag{26}$$

Given these prices, the expected market share of network one reduces to

$$\begin{aligned} d_1(y^{t-1}) &= \frac{1}{2} - \frac{1}{2}\,\frac{K^2n\left(-1 + 2\,y^{\{t-1\}}\right)}{-6\,K^2\delta\,na + 6\,K^2\delta\,n + 3\,K^2n - 2\,\delta\,r_2 - 3\,K^2 + 2\,\delta\,r_2\,K} \\ &= a + (1 - 2a)y^{t-1}. \end{aligned} \tag{27}$$

For the second equality to hold for all values of y^{t-1}, it is necessary to have coefficients of y^{t-1} and 1 to equal. It is easy to verify that both these restriction lead to

$$-\frac{K^2n}{-6\,K^2\delta\,na + 6\,K^2\delta\,n + 3\,K^2n - 2\,\delta\,r2 - 3\,K^2 + 2\,\delta\,r2\,K} - 1 + 2\,a = 0 \tag{28}$$

Also substituting (25) and (26) in the optimization problem of network 1 as given in (0), and computing the coefficients of $(y^{t-1})^2$ on both sides, leads to

$$r_2 + \frac{K^3n^2\left(4\,K^2\delta\,n - \delta\,r_2 + 2\,K^2n + \delta\,r_2\,K - 4\,K^2\delta\,na - 2\,K^2\right)}{\left(-6\,K^2\delta\,na + 6\,K^2\delta\,n + 3\,K^2n - 2\,\delta\,r_2 - 3\,K^2 + 2\,\delta\,r_2\,K\right)^2} = 0. \tag{29}$$

At this point let

$$k(a) = \delta n(2a - 1)^2 + (1 - \delta n - n)(2a - 1) + n.$$

Whenever $n = 0$, it is easy to check that $a = 1/2$ and $r_2 = 0$. For $a \neq 1/2$, solving (28) for r_2 and substituting to (29) with the given definition of $k(a)$, yields $r_2^*(a)$ as given in (2) and a^* solves $g(a)$ as given in (5). Now substituting $r_2(a)$ for the coefficient of y^{t-1} in (25) implies that the coefficient of y^{t-1} in the optimal pricing policy is given by $k^* = k^*(a) = k(a)$. Continuing similarly leads to the solution of the other parameters in equilibrium as given in (1) for l, (3) for r_1 and (4) for r_0. ■

Lemma 2. (Roots of $g(a)$) *There exists a unique root of $g(a)$ in $(0, 1/2)$ and no root of $g(a)$ in $[1/2, 1]$, whenever*

$$n < n_1(\delta, K) < n_0(\delta) < 1.$$

Proof. Define,

$$a_1 = 1 - \frac{1-n}{2\delta n}.$$

The solution of $g(a) = 0$ has to be larger than a_1. Also note that $a_1 < 0$ whenever $n < n_0(\delta)$. In this case,

$$g(a_1) = -\frac{1}{2} \frac{K\left((1-\delta\, n - n)^2 (K-1) + K\,\delta\, n^2\right)}{\delta\,(K-1)\,(1-\delta\, n - n)} < 0.$$

In addition, it is easy to verify that $g(0) \geq 0$ whenever $n < n_1(\delta, K)$.

A few limit that will prove useful are $\lim_{a\to-\infty} g(a) = \infty$, $\lim_{a\to\frac{1}{2}^-} g(a) = -\infty$, $\lim_{a\to\frac{1}{2}^+} g(a) = \infty$ and $\lim_{a\to\infty} g(a) = -\infty$. Thus there are at least two real roots of $g(a)$. But for $n < n_1(delta, K)$, we have $a_1 < 0$ and $g(a_1) < 0$ as well as $g(0) > 0$. Thus, there must be four real roots of $g(a)$. From the limits, it is clear that there must be unique root in $[0, 1/2)$. Furthermore, there is only one root to the right of $1/2$, therefore it is sufficient to check the value of $g(1)$. Substituting $a = 1$ in $g(a)$ yields

$$g(1) = \frac{1}{2} \frac{K^2\,(3-2\,n)}{(K-1)\,\delta} - \frac{1}{2} K\,(1-2\,n)$$

It is easy to verify that $g(1) > 0$, whenever

$$n < n_2(\delta, K) = \frac{1}{2} \frac{3\,K - K\,\delta + \delta}{K - K\,\delta + \delta},$$

as well as $n_1(\delta, K) < n_2(\delta, K)$. Thus, the root must lie to the right of one. ■

Lemma 3. *When there is a unique root of $g(a)$ in $(0,1)$, $0 < k^*(a^*) < \frac{2n}{3}$. This implies that $r_2(a^*) > 0$, $r_1(a^*) > 0$ and $r_0(a^*) > 0$.*

Proof. It is useful for the ease of exposition to rewrite $g(a)$ as

$$g(x) = \frac{1}{2}\,\frac{K^2\,(3\,k(x) - 2\,n)}{(K-1)\,\delta\,x} - \frac{1}{2}\,K\,\left(k(x) - 2\,n\right)x, \tag{30}$$

where $x = (2a-1)$, thus x lies in $[-1,0]$ whenever $a \in [0, \frac{1}{2}]$. It is necessary that $g(x) = 0$ in equilibrium, therefore

$$k^*(x^*) = 2\,\frac{n\,(K - x^2\,(K-1)\,\delta)}{3\,K - x^2\,(K-1)\,\delta}. \tag{31}$$

It is easy to verify that $0 \le k^*(x^*) \le \frac{2n}{3}$ holds whenever $x \in [-1,0]$, $\delta \in [0,1]$, and $K > 1$. Given this, it is straightforward to see $r_2(a^*) > 0$, $r_1(a^*) > 0$ and $r_0(a^*) > 0$. ■

Proposition 1. (Existence of Markov Perfect Equilibrium) *Assume that the conditions of Lemma 2 is satisfied. Then, there exists a symmetric MPE in stationary affine strategies where market is covered and shared each period. The parameters of the affine policy functions are given by Lemma 1. In this equilibrium,*

i. The firm with a higher customer base charges a higher price, i.e $k^*(a^*) > 0$.

ii. The long-run average of market shares are one half, i.e. the after sufficiently many periods market shares oscillate around one half.

iii. There is nonempty set of parameters and stock of old consumers where pricing below cost is the optimal policy.

Proof. Lemmas 2 and 3, proves that there is a nonempty set of parameters where there is a root of $g(a)$ such that $a \in (0, 1/2)$, i.e. market is shared each period and at this root firms have nonnegative nondecreasing value functions. By assumption R is large enough that even in the highest possible price the market will be covered. Thus, the solutions of pricing policy and value function parameters given in Lemma 1 describes a MPE for the relevant parameter set.

By Lemma 3, $k^*(a^*) > 0$, thus the prices are increasing in the customer base of each company. Therefore, the network with larger customer base is able to sustain larger prices in equilibrium.

Observe, that $E[y^{t+1}] = a + (1-2a)E[y^t]$. Thus expected market shares follow a first order difference equation, and this mapping has a fixed point at one half. Using the sample equivalents of the expectations

$$\frac{1}{T+1-t_0} \sum_{t=t_0^{T+1}} y^{t+1} = a + (1-2a)\frac{1}{T-t_0} \sum_{t=t_0^T} y^t$$

and taking the limit as $T \to \infty$, we have the desired property.

When $n = n_1(\delta, K)$, $a = 0$ solves $g(a) = 0$. It is also easy to verify that $g'(0) < 0$, whenever

$$n > n_2(\delta, K) = 2\,\frac{(K-1)\,\delta}{K - 2\,K\,\delta + 5\,K\,\delta^2 - 5\,\delta^2 - \delta} > 0,$$

and there is a nonempty region where $n_2(\delta, K) < n < n_1(\delta, K)$. Hence,in a neighborhood of $n = n_1(\delta, K)$, the solution of $g(a)$ will be very close to zero. Observe that,

$$l^*(0) = -\frac{\delta}{2\,K - K\,\delta^2 + 3\,K\,\delta + \delta^2} < 0,$$

and hence $l^*(a^*)$ will be negative for a^* close enough to zero by continuity of $l^*(a)$ in a. A network with no users will charge price that is simply $l^*(a^*)$, thus a negative or below cost price. ■

Proposition 2. (Long-Run Average of Prices) *The market shares in the long run oscillate around one half. The average prices, $\bar{p}$, satisfy*

$$\max(0, 1 + \delta^2 n - \frac{1}{2}\,\delta - 2\,\delta\,n - n) \le \bar{p} \le 1 - n - \frac{3}{2}\,\delta\,n \le p_{n=0} = 1,$$

whenever $n < n_1(\delta, K)$. Thus, in the presence of network effects long term average of prices are less than what they would have been in the absence of network effects.

Proof. The long run average prices are given by

$$\bar{p}(x) = \frac{1}{2}\,\delta\,k\,(x) + \frac{k\,(x) - n}{x} - \delta\,n, \tag{32}$$

with $x = 2a - 1$. Solving $g(x) = 0$ for $k(x)$ with $g(x)$ as in (30), and substituting in (32) yields

$$\bar{p}^*(x) = -\frac{n\,(2\,\delta\,xK + K + x^2\,(K-1)\,\delta)}{(3\,K - x^2\,(K-1)\,\delta)\,x},$$

which is positive for all $x \in [-1, 0]$. Moreover,

$$\frac{\partial \bar{p}^*(x)}{\partial x} = -\frac{n(\delta^2 x^3\,(K-1)\,(K + x\,(K-1)) - 3\,x^2 K\,\delta\;(\delta\,x + 2) - 3\,K^2\,(1 - 2\,x^2\delta - \delta^2 x^3))}{(3\,K - x^2\delta\;(K-1))^2\,x^2} > 0$$

since $(1 - 2\,x^2\delta - \delta^2 x^3) > 0$, for all $x \in [-1, 0]$ and $\delta \in [0, 1]$. Thus, long-run average prices are increasing in x. Substituting $x = -1$ and $x = 0$ in $\bar{p}(x)$, provides the bounds given in the proposition. ■