Portfolio Selection, Risk Assessment and Heavy Tails*

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Abstract

Distributional assumptions are crucial when assessing the downside risk of a portfolio. Although asset returns are generally heavy–tailed, the normal distribution continues to be the workhorse of theoretical and applied portfolio analysis. We propose a practical approach to risk assessment and portfolio selection which takes heavy–tailedness explicitly into account, while retaining analytical tractability and ease of implementation. An application to a portfolio of stocks shows that the systematic bias in value–at–risk estimation, arising under normality assumptions, is greatly reduced. Furthermore, the excessiveness of the violations of the Basle Committee's standards for model adequacy, which plague the normal model, is largely eliminated.

1 Introduction

Modern portfolio theory assumes that investors and portfolio managers select securities based on the expected return and risk of the resulting portfolios. In the conventional mean–variance framework, portfolio risk is measured in terms of the variance or standard deviation of the portfolio return. In recent practice, necessitated by regulatory requirements as well as internal risk–control strategies, a portfolio's value–at–risk (VaR) has become an important risk measure. The VaR expresses the loss (target level) which the portfolio is expected not to exceed with a given probability (target probability) at the end of a specified holding period. From a statistical viewpoint, the VaR corresponds to a quantile of the profit–and–loss distribution of the portfolio. VaR estimates and, hence, any portfolio–selection decision taking VaR–limits into account, depend on the entertained model for the portfolio's profit–and–loss distribution, which, in turn, depends on the assumption for the joint distribution of the returns of the assets belonging to the portfolio. The most commonly adopted assumption is that the return vectors can be described by a multivariate normal distribution (cf. RiskMetrics Group, 1996). Despite the overwhelming omnipresence of fat tails in empirical return distributions, the popularity of the normal assumption among practitioners persists. Apart from habit, the prevailing use of the normal model has commonly been justified by its analytical tractability. Closure under linear transformation—that is, weighted sums of normally distributed random variables are also normal—, together with the applicability of the central limit theorem make the normal assumption very attractive for theoretical and empirical portfolio analysis.

The family of stable Paretian distributions, of which the normal is a special case, represents a natural generalization of the Gaussian framework. In numerous empirical studies[1] non–Gaussian stable distributions have been found to be much more appropriate for modeling asset returns, while preserving desirable properties of the normal. First, they are closed under linear transformation. Second, they have domains of attraction and are governed by suitable central limit theorems. The former implies that a linear combination of the elements of a stable random vector is again stable; and the latter implies that stable models possess a degree of robustness against misspecifications (cf. Rachev and Mittnik, 2000, p. 2).

Despite of these attractive features, the stable model seems to play no role in practical portfolio analysis. This is somewhat surprising, because the difficulties arising from computational complexities as well as the lack of finite variances, making conventional mean–variance analysis inapplicable, have been largely overcome by now. Computational concerns—due to the fact that there is no general analytic expression for the stable density and distribution functions—are, nowadays, more or less unfounded, given the considerable progress in the computability of stable models during recent years[2] as well as the general access to substantially increased computing power.

Obstacles regarding the theory of stable portfolios analysis had been overcome much earlier with the development of stable mean–variance analogues. Fama (1965b) investigated the distribution of a portfolio of stably distributed assets governed by a single–index structure and subsequently (Fama, 1971) developed a stable version of the CAPM, which obtains efficient portfolios by minimizing the scale parameter of the portfolio–return distribution for a given mean return. Bawa and Lindenberg (1977) and Harlow and Rao (1989) show that, for stable Paretian portfolios, a capital–market equilibrium exists within a mean–lower partial moment framework, and that it is equivalent to that obtained through the mean–scale framework of Fama (1971). Elton, Gruber and Bawa (1979) provide simple portfolio selection rules under stable assumptions.

All these studies on stable portfolio analysis are of theoretical nature. They do not address the problem of *how* to estimate the joint distribution of the individual asset returns. More recent studies by Belkacem, Vèhel and Walter (1995, 2000) and Gamrowski and Rachev (1999) reformulate the approach of Fama in an estimable framework and constitute a first attempt towards empirical analysis. However, the sole empirical focus of these studies is the estimation of the betas, i.e., the stocks' association with an underlying factor, using covariation–based methods, which represent a generalization of the linear–regression framework. They do not address the question of how to estimate the joint distribution of the stocks and the factor, nor do they consider the construction of optimal portfolios from these stocks. As a consequence, they cannot provide any comparisons of portfolio–selection outcomes under Gaussian and non–Gaussian stable assumptions.

In this paper, we exploit certain properties of multivariate stable distributions and develop a practical procedure for estimating multivariate stable models which are governed by multi–factor structures. Specifically, we use the fact that the spectral measure[3] of a factor model has a particular form, which allows us to estimate the distributional parameters along with the factor–association parameters. Having solved the estimation problem, one can then tackle the problem of portfolio optimization and risk assessment in a non–Gaussian stable environment.

In an empirical application to a portfolio of German stocks, we show that risk assessment based on the normal assumption causes severe biases. It tends to overestimate the actual risk associated with high–risk portfolios (i.e., investment strategies associated with high expected returns and less stringent target probabilities) and to underestimate the risk associated with low–risk strategies. Such distortions have crucial consequences for risk management. For example, according to the Capital Adequacy Directive of the Bank of International Settlement in Basle (Basle Committee, 1996a), banks are required to limit their risk exposure in accordance to their trading portfolios' downside risk (measured in VaR terms). Hence, risk–exposure calculations based on the normal assumption may—

depending on the investment strategy—either lead to inefficiencies, because more than the required funds will be held back as a safety cushion and kept from being invested; or they will lead to violations of posted risk limits. As our empirical application demonstrates, the non–Gaussian stable assumption—that is, the relaxation of a single parameter—greatly reduces, if not eliminates the distortions in risk assessment. Moreover, it shows that optimal portfolio weights are greatly affected by the distributional assumption entertained.

The paper is organized as follows. Section 2 presents a brief overview of the properties of multivariate stable distributions. The stable multi–index model is introduced in Section 3; estimation issues are addressed in Section 4. The portfolio selection problem is discussed in Section 5. In Section 6, we apply the approach to a set of stocks belonging to the German DAX index and illustrate consequences for risk assessment, selection decisions and model adequacy. Section 7 concludes.

2 Multivariate Stable Random Vectors

Lacking general closed–form expressions for both the density and distribution function, multivariate stable Paretian distributions—as their univariate counterparts—are commonly defined by their characteristic functions. The logarithm of the joint characteristic function of a stable random vector $Y = (Y_1, \ldots, Y_q)'$ is given by

$$\ln \Phi_\alpha(\theta) = \begin{cases} i(\theta'\mu) - \int_{\mathcal{S}_q} |\theta' s| \left(1 + i\, \frac{2}{\pi}\mathrm{sign}(\theta' s)\right) \ln|\theta' s| \Gamma(ds), & \text{if } \alpha = 1, \\ i(\theta'\mu) - \int_{\mathcal{S}_q} |\theta' s|^\alpha \left(1 - i\ \mathrm{sign}(\theta' s) \tan \frac{\pi\alpha}{2}\right) \Gamma(ds), & \text{if } \alpha \neq 1, \end{cases} \tag{1}$$

where $\alpha \in (0, 2]$ denotes the characteristic exponent (or shape parameter) of the distribution; Γ is a finite measure on the unit sphere, $\mathcal{S}_q$, in $\mathbb{R}^q$; and μ is the location vector in $\mathbb{R}^q$.[4]

In the case of univariate stable Paretian distributions, i.e., $q = 1$ and Y being a scalar, the sphere, $\mathcal{S}_1$, consists of the two points $\{-1, 1\}$. Denoting the probability masses at these points by $\Gamma(-1)$ and $\Gamma(1)$, expression (1) reduces to

$$\ln \Phi_\alpha(\theta) = i\ \theta\mu - |\theta|^\alpha \left[\Gamma(1) + \Gamma(-1) - i\ \mathrm{sign}(\theta)\left[\Gamma(1) - \Gamma(-1)\right] \tan \frac{\pi\alpha}{2}\right]$$

and coincides with the characteristic function of a univariate stable variable, in which case we write $Y \sim S_\alpha(\sigma, \beta, \mu)$, where

$$\sigma = [\Gamma(1) + \Gamma(-1)]^{1/\alpha} \quad \text{and} \quad \beta = \frac{\Gamma(1) - \Gamma(-1)}{\Gamma(1) + \Gamma(-1)}$$

with σ, β, and μ representing the scale, skewness and location parameters, respectively. It follows from the definition of β that if the spectral measure Γ is symmetric, then the scalar Y is symmetrically distributed, i.e, $\beta = 0$ for $\Gamma(1) = \Gamma(-1)$.

If the returns of the assets in a portfolio are characterized by a joint multivariate stable distribution, the aggregate return of the portfolio is given by a linear combination of jointly stable Paretian random variables.

Property 1 (Samorodnitsky and Taqqu, 1994, p. 67) *Let* $w = (w_1, \ldots, w_q)' \in \mathbb{R}^q$ *denote a vector of weights. Then, any linear combination* $w'Y$ *of the components of a stable vector* $Y = (Y_1, \ldots, Y_q)'$ *with spectral measure* $\Gamma(ds)$ *and location vector* μ*, follows the (univariate) stable distribution* $w'Y \sim S_\alpha(\sigma(w'Y), \beta(w'Y), \mu(w'Y))$ *with*

$$\sigma(w'Y) = \left(\int_{\mathcal{S}_q} |w's|^\alpha \Gamma(ds) \right)^{1/\alpha}, \tag{2}$$

$$\beta(w'Y) = \frac{\int_{\mathcal{S}_q} |w's|^\alpha sign(w's) \Gamma(ds)}{\int_{\mathcal{S}_q} |w's|^\alpha \Gamma(ds)}, \tag{3}$$

$$\mu(w'Y) = w'\mu. \tag{4}$$

This result allows us to express the portfolio–return distribution as a function of the multivariate stable distribution of the underlying vector of asset returns. The following property gives rise to a straightforward procedure for modeling asset–return vectors governed by multivariate stable distributions.

Property 2 (Samorodnitsky and Taqqu, 1994, p. 70) *The spectral measure associated with the stable vector* Y *is composed of a finite number of atoms on the unit sphere, if and only if* Y *can be represented by a linear transformation of independent stable random variables.*

There is a natural relationship between Property (2) and the class of index models put forth in portfolio theory; and it is the discreteness of the spectral measure that gives rise to the estimation strategy adopted below.

The next property enables us to derive the spectral measure of a multivariate stable Paretian vector in terms of the spectral measure of a stable random vector with independent elements.

Property 3 (Samorodnitsky and Taqqu, 1994, p. 69) *Let* $X = \big(X_1, \ldots, X_p\big)'$ *with* $X_k \sim S_\alpha(\sigma_k, \beta_k, \mu_k)$*,* $k = 1, \ldots p$*, be a vector of independent random variables with common characteristic exponent* α *(but possibly different scale, skewness and location parameters); and let* $A = \{a_{jk}\}$*,* $j = 1, \ldots, q$*,* $k = 1, \ldots, p$*, be a real matrix. Then, the vector* $Y = (Y_1, \ldots, Y_q)'$ *of linear combination of the independent stable variables* X_k*,* $k = 1, \ldots, p$*, given by*

$$Y = AX,$$

is also stable and has the spectral measure

$$\Gamma \quad = \quad \frac{1}{2}\sum_{k=1}^{p}(\sigma\|a_{\cdot k}\|)^{\alpha}\left[(1+\beta_k)\,\delta\left(\frac{a_{\cdot k}}{\|a_{\cdot k}\|}\right)+(1-\beta_k)\,\delta\left(\frac{-a_{\cdot k}}{\|a_{\cdot k}\|}\right)\right], \tag{5}$$

where $a_{\cdot k}$ *denotes the* k*–th column of matrix* A*;* $\|a_{\cdot k}\| = \left(\sum_{j=1}^{q} a_{jk}^2\right)^{1/2}$ *is the length of the vector* $a_{\cdot k}$*, such that* $a_{\cdot k}/\|a_{\cdot k}\|$ *represents the coordinates of a point on the unit sphere* $\mathcal{S}_q$*; and* $\delta(\cdot)$ *denotes the Dirac–delta function.*

The estimation problem of the stable index model stated below will, by construction, always involve a square transformation matrix A. If A is $d \times d$ with $d = p = q$, it is possible to characterize the joint probability density function of Y in terms of the density of X. Due to the independence of its elements, vector X has the joint density $f_X(x) = \prod_{i=1}^{d} f_{X_i}(x_i)$; and, if A is nonsingular such that $X = A^{-1}Y$, then,

$$f_Y(y) = f_X(A^{-1}y)|\det(A^{-1})|, \tag{6}$$

where $\det(\cdot)$ denotes the determinant of a matrix. Hence, the evaluation of the joint multivariate density of Y only involves the computation of univariate stable densities.

3 The Stable Multi–index Model

A multi-index model establishes the dependence between the returns of different assets through a set of common market factors. Each return series evolves as a linear combination of these factors, but is perturbed by an additive idiosyncratic noise process. Formally, if there are N assets with returns $R_i,\ i = 1,\dots,N$, and M factor returns $F_j,\ j = 1,\dots,M$, then, in any given period, the return of asset i is given by

$$R_{it} = \mu_{R_i} + \sum_{j=1}^{M} b_{ij}(F_{jt} - \mu_{F_j}) + \varepsilon_{it}, \quad i = 1,\dots,N, \tag{7}$$

where ε_{it} denotes the idiosyncratic disturbance; μ_{F_j} is the mean of factor j; and b_{ij} reflects the systematic influence of factor j on asset i. In matrix notation, the N equations in (7) can be written as

$$R_t = \mu_R + B(F_t - \mu_F) + \varepsilon_t,$$

where $R_t = (R_{1t},\dots,R_{Nt})'$, $\mu_R = (\mu_{R_1},\dots,\mu_{R_N})'$, $F_t = (F_{1t},\dots,F_{Mt})'$, $\mu_F = (\mu_{F_1},\dots,\mu_{F_M})'$, $\varepsilon_t = (\varepsilon_{1t},\dots,\varepsilon_{Nt})'$, and $B = \{b_{ij}\},\ (i = 1,\dots,N,\ j = 1,\dots,M)$.

Under the usual normality assumption, the disturbances and factors follow independent univariate normal distributions, that is, in present notation, $\varepsilon_{it}, F_{jt} \sim S_2(\cdot,\cdot,\cdot)$. In our analysis, we relax the normality assumption by allowing the disturbances and factors to follow independent heavy-tailed stable distributions, i.e., $\varepsilon_{it}, F_{jt} \sim S_\alpha(\cdot,\cdot,\cdot)$, with $0 < \alpha \leq 2$. Thus, in a symmetric setting, we generalize the normal multi–index model by simply relaxing a single parameter. That is, instead of imposing the Gaussian restriction $\alpha = 2$, we allow $\alpha \in (0, 2]$. A second parameter is added, namely $\beta \in [-1, 1]$, if we allow for asymmetry.

Assumption 1 *The random variables* $\{\varepsilon_{1t}, \ldots, \varepsilon_{Nt}, F_{1t}, \ldots, F_{Mt}\}$ *are independent and follow (univariate) stable distributions with a common shape parameter* $0 < \alpha \leq 2$*, that is,* $\varepsilon_{it} \sim S_\alpha(\sigma_{\varepsilon_i}, \beta_{\varepsilon_i}, 0)$ $(i = 1, \ldots, N)$ *and* $F_{jt} \sim S_\alpha(\sigma_{F_j}, \beta_{F_j}, \mu_{F_j})$ $(j = 1, \ldots, M)$*.*

According to Assumption 1, the return on an individual asset, R_{it}, is a linear combination of $M + 1$ independent stable random variables, namely the M factors plus the idiosyncratic disturbance ε_{it}. Combining return and factor vectors, we have

$$\begin{bmatrix} R_t \\ F_t \end{bmatrix} = \begin{bmatrix} \mu_R \\ \mu_F \end{bmatrix} + \begin{bmatrix} I_N & B \\ \mathbf{0} & I_M \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varphi_t \end{bmatrix}, \tag{8}$$

where $\varphi_t = F_t - \mu_F$ denotes the mean–corrected factor vector. Defining

$$A := \begin{bmatrix} I_N & B \\ \mathbf{0} & I_M \end{bmatrix}$$

and noting that (8) is a linear transformation in the sense of (6) with

$$A^{-1} = \begin{bmatrix} I_N & -B \\ \mathbf{0} & I_M \end{bmatrix} \tag{9}$$

and $\det(A^{-1}) = 1$, we can use Property 3 to construct the spectral measure and the location vector for $(R_t', F_t')'$. Then, in accordance with (3), the spectral measure of the return vector generated by the M-factor model is given by

$$\begin{aligned} \Gamma \; &= \; \frac{1}{2}\sum_{k=1}^{N} \sigma_{\varepsilon_k}^{\alpha} \left[(1+\beta_{\varepsilon_k})\delta(a_{\cdot k}) + (1-\beta_{\varepsilon_k})\delta(-a_{\cdot k})\right] \qquad (10) \\ &\quad + \frac{1}{2}\sum_{k=1}^{M} (\sigma_{F_k}\|a_{\cdot N+k}\|)^{\alpha} \left[(1+\beta_{F_k})\,\delta\!\left(\frac{a_{\cdot N+k}}{\|a_{\cdot N+k}\|}\right) + (1-\beta_{F_k})\,\delta\!\left(\frac{-a_{\cdot N+k}}{\|a_{\cdot N+k}\|}\right)\right], \end{aligned}$$

with $\|a_{\cdot N+k}\| = (1 + \sum_{i=1}^{N} b_{jk}^2)^{1/2}$ and

$$\mu = \begin{bmatrix} \mu_R \\ \mu_F \end{bmatrix}. \tag{11}$$

being the location vector of $(R_t', F_t')'$.

The following proposition summarizes the above results.

Proposition 1 *Let $(R_t', F_t')'$ be a vector of asset and factor returns generated by the multi–index model (7). If Assumption 1 holds, then $(R_t', F_t')'$ follows a multivariate stable law whose spectral measure and location vector are given by (10) and (11), respectively.*

Once the parameters of the stable index model have been specified, i.e., given (estimates of) Γ and μ, Property 1 allows us to straightforwardly calculate the implied distribution of any n–asset portfolio, $n \leq N$, which can be constructed from the N assets.

4 Estimation

To estimate the parameters of the stable M–factor model we use (9) to rearrange (8), such that

$$\begin{bmatrix} \varepsilon_t \\ \varphi_t \end{bmatrix} = \begin{bmatrix} I_N & -B \\ \mathbf{0} & I_M \end{bmatrix} \left(\begin{bmatrix} R_t \\ F_t \end{bmatrix} - \begin{bmatrix} \mu_R \\ \mu_F \end{bmatrix} \right) . \tag{12}$$

Assumption 1 implies that the vector $(\varepsilon_t', \varphi_t')'$ has independent components with stable densities $f_{\varepsilon_{it}}(\cdot; \sigma_{\varepsilon_i}, \beta_{\varepsilon_i}, 0)$ and $f_{\varphi_{jt}}(\cdot; \sigma_{\varphi_{jt}}, \beta_{\varphi_{jt}}, 0)$. Given the return and factor realizations $R_t = (R_{1t}, \ldots, R_{Nt})'$ and $F_t = (F_{1t}, \ldots, F_{Mt})'$ for period t, the value of the joint density for the t-th observation is given by

$$\begin{aligned} f_{\varepsilon_t, \varphi_t}(R_t, F_t; \theta) &= \left(\prod_{i=1}^{N} f_{\varepsilon_{it}} \left(R_{it} - \sum_{k=1}^{M} b_{ik} F_{kt} - \mu_R; \theta_{\varepsilon_i} \right) \right) \\ &\quad \times \left(\prod_{j=1}^{M} f_{\varphi_{jt}} (F_{jt} - \mu_{F_j}; \theta_{\varphi_j}) \right) \\ &=: \; f_{\varepsilon_t} \times f_{\varphi_t}, \end{aligned}$$

where $\theta_{\varepsilon_i} = (\sigma_{\varepsilon_i}, \beta_{\varepsilon_i}, \mu_{R_i})'$ and $\theta_{\varphi_j} = (\sigma_{\varphi_j}, \beta_{\varphi_j}, \mu_{F_j})'$.

The vector $\theta = (b', \theta_\varepsilon', \theta_\varphi', \alpha)'$, with $b = \text{vec}(B)$,[5] $\theta_\varepsilon = (\theta_{\varepsilon_1}', \ldots, \theta_{\varepsilon_N}')'$ and $\theta_\varphi = (\theta_{\varphi_1}', \ldots, \theta_{\varphi_M}')'$, collects all $MN + 3(M+N) + 1$ parameters of the stable index model Given T observations and defining the $N \times T$ matrix $\mathbf{R} = (R_1, \ldots, R_T)$ and the $M \times T$ matrix $\mathbf{F} = (F_1, \ldots, F_T)$, the joint density of $\mathbf{R}$ and $\mathbf{F}$ is given by

$$f_{\varepsilon,\varphi}(\mathbf{R}, \mathbf{F}; \theta) = \prod_{t=1}^{T} (f_{\varepsilon_t} \times f_{\varphi_t}).$$

The maximum likelihood (ML) estimator of parameter vector θ is obtained by maximizing the log–likelihood function

$$\mathcal{L}(\theta; \mathbf{R}, \mathbf{F}) \propto \sum_{t=1}^{T} (\log f_{\varepsilon_t} + \log f_{\varphi_t})$$

with respect to θ.

In practical applications, the large number of parameters to be estimated, namely, $MN + 3(M + N) + 1$, may render the ML estimator infeasible, due to the high computational complexity that arises. An alternative and practically more feasible estimation strategy consists of a combination of ordinary least squares (OLS) estimation of the b_{ij} coefficients and ML estimation of the distributional parameters.[6] The consistency of the OLS estimates of b_{ij} in this setting where regressors are also α-stable distributed is established in Kurz-Kim, Rachev and Samorodnitsky (2004). Also, for $1 < \alpha \leq 2$, which appears to hold for financial applications, the mean vectors μ_R and μ_F are consistently estimated by the sample means $\bar{R}_t$ and $\bar{F}_t$, respectively. Then, the ML estimator can be used to estimate the remaining distributional parameters, that is, $\sigma_{\varepsilon_i}, \sigma_{\varphi_j}, \beta_{\varepsilon_i}, \beta_{\varphi_j}$, and α. Specifically, given the OLS estimate of B, we transform the de–meaned data, as implied by (12), and perform a joint ML estimation of the $M + N + 1$ parameters σ_{ϵ_i}, σ_{φ_j} and α from $(\varepsilon_t', \varphi_t')'$, $t = 1, \ldots, T$. Doganoglu and Mittnik (2002) analyze the performance of this combined estimation procedure and find that it performs well both in terms of convergence, speed and accuracy.

5 Portfolio Selection and Risk Assesment

Because the standard mean–*variance* approach is not applicable to non–Gaussian stable portfolios, Fama (1971) and Bawa and Lindenberg (1977) develop a mean–*scale* analogue. For the expected portfolio return to be finite, they assume that $\alpha \in (1, 2]$. This assumption is justified on empirical grounds and will be adopted in the following.[7]

Letting μ_p and σ_p, respectively, denote the expected mean and the scale parameter of the portfolio return, the set of efficient portfolios is derived by finding the weight vector $w = (w_1, \ldots, w_N)'$ which solves the optimization problem:

$$\max_w \mu_p = w'\mu \tag{13}$$

subject to

$$\sigma_p \leq \sigma_p^*,$$

$$w'\mathbf{1}_N = 1,$$

and, if short–selling is not allowed,

$$w_i \geq 0, \qquad i = 1, \ldots, N,$$

where σ_p^* is the risk limit; and $\mathbf{1}_N$ denotes an $N \times 1$ vector of ones. For $\alpha < 2$, relationship (2) is used to compute the portfolio scale. In the normal case, i.e., $\alpha = 2$, we use the portfolio standard deviation $\sigma_p = (w'\Sigma w)^{1/2}$, where Σ denotes the covariance matrix implied by the normal index model, instead of (2).

Given that investors base their decisions solely on expected risk–return considerations, the decision problem can be placed in any suitable expected return–risk space. In the mean–variance framework, the minimum–variance set reflects the minimum–risk portfolios—with risk being measures in terms of the portfolio variance or standard deviation—that are associated with feasible expected portfolio returns. To handle the stable case, Fama (1971) and Bawa and Lindenberg (1977) adopt the mean–scale framework. Unfortunately, scale parameters of stable distributions with different characteristic exponents cannot be meaningfully compared. Therefore, the implications of different stable assumptions (here, $\alpha = 2$ versus $\alpha < 2$) are not easily accessible in this framework. However, by using a portfolio's VaR as a risk measure and translating the decision problem to the mean–VaR space, we can examine the consequences of alternative distributional assumptions in a common risk–return setting. For a given target probability, ν, there is a strictly affine one–to–one correspondence between the scale parameter, σ_p, and VaR_ν–level (i.e., the negative $100 \times \nu\%$–quantile) of the stable profit–and–loss distribution.[8] Therefore an optimization will yield the same weight vector, regardless of whether it is conducted in a mean–scale or a mean–VaR_ν framework.

An efficient portfolio in the mean–VaR plane is given by solving the optimization problem:

$$\max_w w'\mu_R \tag{14}$$

subject to

$$\mathrm{VaR}_\nu \leq \mathrm{VaR}_\nu^*,$$

$$w'\mathbf{1}_N = 1,$$

and, if there is no short–selling,

$$w_i \geq 0, \qquad i = 1, \ldots, N,$$

where VaR_ν^* denotes the specified VaR–limit with target probability ν. Note that investment strategies based on optimization (14) amount to adopting Telser's (1955) criterion for portfolio selection.

In the next section we employ the mean–VaR framework to empirically illustrate the consequences for portfolio selection and risk assessment, when taking the observed heavy–tailedness of asset returns into account.

6 An Application to DAX Stocks

6.1 The Data

We consider portfolios constructed from a set of 26 stocks belonging to the German DAX index. The sample consists of $T = 1828$ daily observations of (dividend–corrected) returns on the 26 stocks and the composite DAX (CDAX), a broad market index, covering the period from January 1991 to April 1998.

The names of the stocks and various summary statistics, are listed in Table 1. For all stocks the sample kurtosis exceeds 3, the value compatible with the normal assumption; about half of them do so significantly at the 95% level. The hypothesis of joint normality is rejected even more strongly, namely at the 99% level, when testing for excess kurtosis of the joint distribution of the 26 stocks. The hypothesis of symmetry is also rejected at the 99% level both individually and jointly. However, an inspection of the skewness statistics for individual stocks reveal that the asymmetries of the return distributions are not extreme, albeit significant.

Table 1 somewhere here.

6.2 Estimation Results

Assuming that the dependence of the 26 stock–return series can be captured by a single–index model, with the CDAX being the underlying factor, we estimate the model parameters under both the normal and (non–Gaussian) stable assumption. For the latter we estimate both a symmetric ($\beta = 0$) and an asymmetric ($\beta \in [-1, 1]$) version. The normal model is estimated by the usual sample means, variances and covariances. For the stable model we employ the combined estimation procedure described in the previous section.

The estimation results are reported in Table 2. There, to take the different scaling conventions for normal and stable random variables into account, namely $S_2(\sigma, \cdot, \cdot) \equiv \mathrm{N}(\cdot, 2\sigma^2)$, we divide the standard-deviation estimates for the normal by $\sqrt{2}$.

Table 2 somewhere here.

An evaluation of the log–likelihood functions with the parameter estimates favors the stable models. The values of the standard likelihood–ratio test statistic for normal versus both symmetric and asymmetric stable models, given by

$$LR_{N,SS} = -2(\text{Loglik}_{normal} - \text{Loglik}_{sym.stable}) = 6987.1$$

and

$$LR_{N,AS} = -2(\text{Loglik}_{normal} - \text{Loglik}_{asym.stable}) = 7056.2,$$

respectively, by far exceed 6.635, the 99%–critical value of the χ_1^2 distribution and, thus, clearly reject the null hypothesis "$\alpha = 2$". Moreover, the likelihood–ratio test statistic for symmetric versus asymmetric stable models, given by

$$LR_{SS,AS} = -2(\text{Loglik}_{sym.stable} - \text{Loglik}_{asym.stable}) = 69.12,$$

strongly favors the asymmetric over the symmetric multivariate stable Paretian distribution. The estimate of the overall shape parameter, $\hat{\alpha} = 1.705$ (1.706) for the asymmetric (symmetric) stable model, with an approximate standard error of 0.003, also implies rejection of the normal assumption of $\alpha = 2$.

6.3 Comparison of Estimated and Empirical Frontiers

Although the data strongly favor the non-Gaussian stable over the Gaussian model, this does not necessarily imply that risk assessment and portfolio selection have to be greatly affected by adhering to the Gaussian assumption. To examine this question, we derive the efficient mean–VaR frontiers from both the estimated normal and stable models. Let $\widehat{\text{VaR}}^{model}_{\nu,\mu^*}$ denote the VaR–levels (associated with the target probability ν and the expected portfolio return μ^*) implied by the estimated model under consideration. To compare the empirical validity of the estimated frontiers, we also derive the corresponding *empirical* frontiers. Letting w_{μ^*} denote the weight vector that is associated with the (minimum–risk) expected portfolio return μ^*, the negative value of the $[\nu T]$–th order statistic, denoted by $\text{VaR}^{emp}_{\nu,\mu^*}$, of vector $w'_{\mu^*}\mathbf{R}$, represents the empirical counterpart to the estimates $\widehat{\text{VaR}}^{model}_{\nu,\mu^*}$. Here, $[\nu T]$ stands for the smallest integer not exceeding the product νT; and $\mathbf{R}$, the empirical returns matrix was defined in Section 4.

Figure 1 shows the efficient mean–VaR frontiers for the six target probabilities $\nu = 0.01, 0.025, 0.05, 0.075, 0.1, 0.25$. Note that, due the discreteness of the empirical return distributions, the empirical minimum–VaR frontiers (dotted lines) are—especially for low ν values—somewhat rugged and not necessarily concave.

Figure 1 somewhere here.

Visual inspection of the plots in Figure 1 reveals that the stable models provide the better overall fit. The normal frontiers (dash–dot lines) systematically deviate from the empirical frontiers. They underestimate the risk for the $\nu = 0.01$ target probability and then wander to the right of the empirical frontiers as ν increases, so that the normal overestimates the risk for $\nu \geq 0.05$. Moreover, the bias increases as the VaR–limit increases. The restriction to normality, i.e. $\alpha = 2$, induces a clockwise rotation of the estimated frontiers relative to the empirical ones. The stable frontiers (solid lines) approximate the empirical frontiers much more closely. In cases where the model is not so perfect, namely for $\nu = 0.25$ and $\nu = 0.01$, the stable frontiers are to the right of the empirical frontier, implying that the stable framework tends to provide a somewhat more conservative risk assessment.

The dominance of the stable model ceases somewhat for the smallest target probability, $\nu = 0.01$. However, in view of the sample size $T = 1828$, the results for these probability levels have to be treated more cautiously, because they rely on estimates of a rather low–order statistic, namely the 18^{th}, of the vector $-w'R$ and is, thus, more error prone. The ruggedness of the empirical frontiers indicates already that a smaller number of tail observations makes results less reliable.

We numerically summarize the overall fit of a model's implied frontier estimates by evaluating the following aggregate goodness-of-fit measures: the *mean squared deviation*

$$\mathrm{MSD}_\nu = \frac{1}{n}\sum_{k=1}^{n}\left(\mathrm{VaR}^{emp}_{\nu,\mu_k^*} - \widehat{\mathrm{VaR}}^{model}_{\nu,\mu_k^*}\right)^2, \tag{15}$$

where μ_k^* denotes the kth grid point at which we evaluate the frontiers, and n the number of grid points; the *mean absolute deviation*

$$\mathrm{MAD}_\nu = \frac{1}{n}\sum_{k=1}^{n}\left|\mathrm{VaR}^{emp}_{\nu,\mu_k^*} - \widehat{\mathrm{VaR}}^{model}_{\nu,\mu_k^*}\right|; \tag{16}$$

and the *mean deviation*

$$\mathrm{MD}_\nu = \frac{1}{n}\sum_{k=1}^{n}\left(\mathrm{VaR}^{emp}_{\nu,\mu_k^*} - \widehat{\mathrm{VaR}}^{model}_{\nu,\mu_k^*}\right). \tag{17}$$

The empirical results for these measures, using $n = 16$ grid points, are given in Tables 3. The stable models outperform the normal model for most of the target probabilities and measures considered. One exception is the symmetric stable distribution. Its fit for target probability $\nu = 0.01$ is slightly worse than that of the normal. Nevertheless, as noted before, the results for this probability level should be interpreted with more caution in view of the relatively small sample size The asymmetric stable distribution, on the other hand, clearly dominates the normal for all target probabilities and goodness-of-fit measures considered. The sole exception occurs for the MD measure at $\nu = 0.025$. Here, as Figure 1 shows, the normal frontier—although systematically biased—intersects the empirical frontier in such a fashion that the average error is close to zero.

Table 3 somewhere here.

Next, we consider the implications of the distributional assumption for portfolio selection. Restricting ourselves to the—in practice particularly relevant—target probability

$\nu = 0.01$, we derive the weights of the VaR–efficent portfolios for $\text{VaR}_{.01}$–limits 2, 3, 4, and 5 under normal and stable assumptions. The weights, shown in Table 4 together with the empirically realized VaR–levels, indicate that the optimal portfolios selected for the specified VaR–constraint depend crucially on the assumed distribution both in terms of composition and risk assessment. Under the stable assumption a portfolio manager would correctly conclude that, for the given sample, it is not possible to construct a portfolio satisfying a $\text{VaR}_{.01}$–limit of 2, because this limit falls below the estimated global minimum $\text{VaR}_{.01}$–risk, which is above 2.5 for the stable models. According to the estimated normal model, a $\text{VaR}_{.01}$–limit of 2 should be feasible. However, the empirically realized $\text{VaR}_{.01}$–level for the resulting portfolio is 2.55 and exceeds the target VaR–limit by about 27.5%. Also, for the VaR–limits 3, 4 and 5 the normal model selects portfolios, which violate the specified limits. No such violations occur for the stable models.

These empirical findings show that the this-tailedness due to the normal assumption results in portfolios which generally underestimate the portfolio risk as measured by $\text{VaR}_{.01}$.

Table 4 somewhere here.

The distributional assumptions also induces strikingly different portfolios. The stable models lead to portfolios which are considerably more diversified. For the $\text{VaR}_{.01}$–levels 3, 4 and 5, the normal selects from the 26 assets only 6, 3 and 3 assets, respectively, whereas the optimal stable portfolios consist of 16, 10 and 6 stocks, respectively. Also the portfolio weights differ substantially. For example, for the $\text{VaR}_{.01} = 3$ target the optimal normal portfolio allocates 45% to the high–risk and high–return stock SAP, whereas the stable ones only about 19%. On the other hand, Veba, the asset with the least risk in terms of standard deviation (see Table 1) is favored more by the stable models. For the $\text{VaR}_{.01}$–level 3, 4 and 5, the stable models allocate about 17, 15 and 5% to Veba, whereas the normal model assigns 16, 0 and 0%, respectively. Overall, it is evident that the normal model overestimates the gains of diversification and, thus, selects fewer assets and tends to allocate more weight to high–risk assets.

The portfolio weights for the symmetric and asymmetric stable models are very similar. This may suggest that the portfolio selection itself is not so much affected by assuming symmetry. However, as is indicated by the goodness-of-fit measures (see Table 3) for $\nu = 0.01$ and the backtesting results reported next, risk assessment based on $\text{VaR}_{.01}$ measures improves when asymmetry is accounted for.

6.4 Backtesting According to the Basle-Committee Framework

So far, we have examined the appropriateness of the models in terms of statistical criteria by measuring their fit to the data and the distance between the estimated and empirical frontiers. According to the Basle Accord, banks using an internal model to assess the market risk of trading positions have to continually monitor the adequacy of the model by performing certain "backtesting" procedures. The accuracy of the model is judged by the number of "exceptions", that is, the number of times the realized loss of a position exceeds the VaR–estimate generated by the model. Specifically, to judge the model's reliability, national oversight authorities should focus on the number of $\mathrm{VaR}_{.01}$–limit violations for a one–day holding period which occur during the preceding 250–day period (cf. Basle Committee, 1996b). If an excessive number of such violations is observed, banks may have to increase their capital requirements or may even be disallowed to use the model.

Theoretically, over a period of τ days, a correctly specified model should exceed the VaR_{ν}–limit on $\nu\tau$ days. Thus, on average, there should be 2.5 violations of the $\mathrm{VaR}_{.01}$–limit over a period of 250 days. The Basle Committee has set practical standards for evaluating the adequacy of a model by specifying three "zones" for the violation frequency over a 250–day horizon:

Green zone (0–4 violations): The model is considered to be accurate, requiring no supervisory response.

Yellow zone (5–9 violations): The model is more likely to be inaccurate than accurate. Capital requirements should be increased by raising the multiplication factor from 3 to somewhere between 3.4 and 3.85.[9]

Red zone (more than 9 violations): There is a presumption that the model is flawed. The bank's capital requirements should be automatically increased by raising the multiplication factor from 3 to 4. Moreover, the bank should be required to work on improving the model immediately.

To examine how the normal and stable models fair under the Basle Committee adequacy standards, we subjected them to the backtesting exercises using the three–zone approach. We divided the sample into seven non-overlapping subsamples, comprised of 250 days each.[10] We, then, counted how often each model violates the $\mathrm{VaR}_{.01}$–limits 2, 3, 4, and 5 in each of the seven intervals.

The results, given in Table 5, show that the normal model has a higher violation frequency. For the limit $\mathrm{VaR}_{.01} = 2$, it falls two out of seven times into the green zone; it is once in the red, and four times in the yellow zone. Recall that, for $\mathrm{VaR}_{.01} = 2$, there are no results for the stable models, because the limit falls below the global minimum–risk level implied by those models. For $\mathrm{VaR}_{.01} = 3$, the normal model ends up four times in the green zone and three times in the yellow; while the stable models fall six times into the green and once into the yellow, respectively. The patterns are not much different for the other limits. Over all $\text{VaR}_{.01}$–limits considered, the normal model causes altogether 13 violations (12 yellow and 1 red), whereas the stable ones lead only to 2 yellow violations each. Aggregating over all seven subsamples, the normal model has about six times as many violations as the stable models. For the normal, the averages range from 2.86 to 5 and, thus, exceed the critical target number of 2.5 for all $\text{VaR}_{.01}$–levels considered. The stable models, on the other hand, provide a fairly close match; the average numbers of violations ranging from 1.71 to 1.86 and staying below the critical value 2.5.

Table 5 somewhere here.

The exercise just described is not realistic in the sense that, in practice, backtests ought to be done much more frequently than just once every 250 days. Figure 2 plots, for each of the $\text{VaR}_{.01}$–level considered, the number of violations obtained when performing backtests on a daily basis. The horizontal reference lines below the values 5 and 10 divide the range into the green, yellow and red zones. The plots reveal that the normal model generates risk assessments, which most of the time fall into the green zone, but also disturbingly often into the yellow and red zones. The stable models stay most of the time in the green zone; they end up in the (lower range of the) yellow zone about as many times as the normal does in the red; and they never fall into the red zone.

Figure 2 somewhere here.

7 Conclusion

Although it has been widely recognized that asset returns tend to be heavy–tailed, commonly adopted procedures for assessing portfolio risk rely on the assumption of a joint normal distribution for the underlying assets. This seems to be mainly due to the difficulties that arise in the context of estimation and portfolio optimization when heavy–tailed distributions are assumend to describe asset returns. In this paper, we have presented a convenient approach for estimating the parameters of portfolios governed by a multivariate non–Gaussian stable distribution and a multi–factor structure. It gives rise to a procedure for optimal portfolio selection under stable assumptions which can be easily implemented in practice. The question of how to determine the appropriate (number of) factors in the presence of heavy–tailed returns has not been addressed here. This is—even in more standard settings (see, for example, Dhrymes, Friend and Gultekin, 1984; and Brown, 1989)—a nontrivial problem and will be addressed in future work.

Focusing on stocks belonging to the German DAX index, we have examined the accuracy in risk assessment using VaR as downside–risk measure. The application shows that the Gaussian one–factor model leads to systematically distorted risk-return frontiers and, thus, risk assessments. The distortions are substantially reduced when assuming that returns follow a symmetric stable non–Gaussian distribution, that is, by relaxing only a single model parameter, the characteristic exponent α. A further, though less dramatic, improvement is achieved by relaxing a second parameter to allow also for asymmetries in the return distributions.

Backtesting analyses, which follow the Basle Committee's three–zone approach for evaluating a model's adequacy for risk assessment, reveal that the normal assumption causes excessive violations of the prespecified VaR–limits at the 1% level, whereas the non–Gaussian stable models generate, on average, permissible numbers of violations. The stable model leads to $\text{VaR}_{.01}$–optimal portfolios that are slightly on the conservative side. This would induce slight inefficiencies in the allocation of funds due to larger amounts of capital being held back as a safety cushion. However, these inefficiencies are dwarfed by those arising from the normal assumption's excessive frequency of failing the Basle Committee's standards inducing increased capital requirements.

Besides the differences in risk assessment, the application demonstrates that portfolios optimized under the stable assumption differ quite substantially from those chosen under the normal. The former tend to be more diversified, i.e., the investments are spread over more assets and somewhat more evenly across the assets entering the portfolio. The generalization from symmetric to asymmetric stable assumption does not affect portfolio selection significantly, but it leads to more accurate risk assessments by providing more reliable efficient mean-VaR frontiers.

Clearly, one does not expect that asset returns truly follow a stable distribution. But the empirical results suggest that the stable non–Gaussian model for portfolio analysis provides a more realistic working hypothesis for the empirical risk–return characteristics and, thus, for portfolio–selection and risk–assessment decisions—while still preserving analytical tractability.

Notes

[1]See, for example, Fama (1965a), Akgiray and Booth (1989), Mittnik and Rachev (1993), and McCulloch (1997), Rachev and Mittnik (2000), and references therein.

[2]See, for example, Doganoglu and Mittnik (1998), McCulloch (1998), Mittnik, Doganoglu and Chenyao (1999), Mittnik, Rachev, Doganoglu, and Chenyao (1999), and Nolan (1999).

[3]The spectral measure defines the dependence structure of multivariate stable vectors (see Section 2 below).

[4]The subsequent discussion of the properties of the multivariate stable Paretian distributions closely follows that of Samorodnitsky and Taqqu (1994).

[5]Here, vec$(\cdot)$ denotes the vectorization operator, which stacks all columns of a matrix into one column vector.

[6] Blattberg and Sargent (1973) show that the coefficients of a regression model with stable disturbances can be consistently estimated via OLS. However, our setup differs from their's in that we have a stochastic regressors (see Doganoglu and Mittnik, 2002).

[7]If $\alpha \leq 1$, the analysis would have to be placed in a *location*–scale framework.

[8]By equating the VaR_ν–level with the (negative) $100 \times \nu\%$–quantile of the return distribution, the analysis assumes an initial investment of unity and, thus, is independent of the size of the initial investment.

[9]The specific increments with which the multiplication factor ought to be raised depends on the violation frequency. In fact, the guidelines state that "[t]he concern about 'fat tails' was also an important factor in the choice of the specific increments..." (Basle Committee, 1996b, p. 8).

[10]The first subsample contains observations 1 through 250, the second 251 through 500, etc. up to the seventh ranging 1501 through 1750.

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Table 1: List of Stocks and Summary Statistics

Name	Label	Mean	Standard Deviation	Skewness	Kurtosis
CDAX	CDAX	0.0588	0.8150	-0.9171	12.3962
Allianz	ALZ	0.0670	1.4119	0.0519	7.5104
Babcock	BAB	-0.0191	2.6953	-2.3772	41.0101
BASF	BAS	0.0919	1.3461	-0.2105	5.4588
Bayer	BAY	0.0861	1.3508	-0.2336	6.6947
BMW	BMW	0.1057	1.5189	-0.1374	10.7971
Commerzbank	COM	0.0788	1.3912	-0.2018	15.5800
Conti	CON	0.0560	1.6739	-0.2250	6.4051
Degussa	DGX	0.0785	1.5289	0.1647	6.6580
Deutsche Bank	DBK	0.0571	1.2972	-0.1745	9.1049
Dresdner Bank	DRD	0.0712	1.3241	-0.1935	10.5195
Henkel	HEN	0.0568	1.3818	-0.6541	11.1981
Karstadt	KAR	0.0278	1.3928	-0.2072	8.2864
Linde	LIN	0.0341	1.2683	-0.5505	11.9762
MAN	MAN	0.0423	1.5172	-0.6600	14.8864
Mannesman	MMN	0.1037	1.5545	-0.4986	11.9634
Metallges.	MTL	-0.0321	2.4671	-0.7591	21.7434
Muencher Rueck	MUE	0.0817	1.4522	0.1557	8.6872
Preussag	PRS	0.0600	1.3769	-0.3261	8.2573
RWE	RWE	0.0624	1.2591	-0.3664	12.3564
SAP	SAP	0.2217	2.2700	-1.0426	20.2714
Schering	SCH	0.0625	1.3630	-0.1284	5.8942
Siemens	SIE	0.0413	1.2306	-0.5680	11.5163
Thyssen	THY	0.0574	1.5292	-0.2728	8.1100
Veba	VEB	0.0878	1.2034	-0.2068	9.8004
Viag	VIA	0.0679	1.2470	0.0745	14.1394
VW	VSW	0.0875	1.7186	-0.3618	8.7667

The data sample consists of 1829 observations covering the period from January 1991 to April 1998. The sample skewness and sample kurtosis are computed by $\left(\sum_{t=1}^{T}(R_{it}-\bar{R}_i)^3\right)\left(\sum_{t=1}^{T}(R_{it}-\bar{R}_i)^2\right)^{-\frac{3}{2}}$ and $\left(\sum_{t=1}^{T}(R_{it}-\bar{R}_i)^4\right)\left(\sum_{t=1}^{T}(R_{it}-\bar{R}_i)^2\right)^{-2}$, respectively, where $\bar{R}_i$ denotes the sample mean for the i-th stock.

Table 2: Estimation Results

Asset i	μ_i	$b_{i,M}$	Normal $\sigma_{\varepsilon_i}/\sqrt{2}$	Sym. Stable σ_{ε_i}	Asym. Stable σ_{ε_i}
CDAX	0.059	1	0.576	0.465	0.465
ALZ	0.067	1.358	0.620	0.532	0.531
BAB	-0.019	1.230	1.769	1.083	1.080
BAS	0.092	1.241	0.628	0.539	0.537
BAY	0.086	1.270	0.614	0.524	0.524
BMW	0.106	1.304	0.767	0.594	0.592
COM	0.079	1.176	0.713	0.557	0.556
CON	0.056	1.100	1.000	0.840	0.837
DGX	0.078	1.116	0.869	0.739	0.737
DBK	0.057	1.232	0.581	0.475	0.475
DRD	0.071	1.170	0.649	0.497	0.496
HEN	0.057	1.033	0.775	0.619	0.618
KAR	0.028	0.958	0.816	0.696	0.694
LIN	0.034	1.037	0.669	0.553	0.552
MAN	0.042	1.194	0.823	0.684	0.683
MMN	0.104	1.211	0.849	0.690	0.689
MTL	-0.032	1.124	1.620	0.974	0.974
MUE	0.082	1.026	0.840	0.681	0.681
PRS	0.060	0.990	0.789	0.658	0.657
RWE	0.062	1.095	0.628	0.501	0.500
SAP	0.222	1.105	1.473	1.074	1.070
SCH	0.062	0.951	0.793	0.631	0.630
SIE	0.041	1.177	0.545	0.432	0.432
THY	0.057	1.174	0.844	0.683	0.681
VEB	0.088	1.066	0.589	0.487	0.486
VIA	0.068	0.938	0.697	0.567	0.567
VSW	0.087	1.439	0.888	0.734	0.735
α			2	1.7062	1.7045
β			0	0	0.1213
Log-Likelihood			-75350.4	-71856.9	-71822.3

Because the stable distribution with $\alpha = 2$, i.e., $S_2(\sigma, \cdot, \cdot)$, is equivalent to the normal distribution $N(\cdot, 2\sigma^2)$. We divide the standard–deviation estimates for the normal by $\sqrt{2}$ to make the scales of the two distributions more compatible.

Table 3: Summary Measures for the Goodness-of-Fit of Estimated Efficient Frontiers

Target Probability ν	Mean Squared Deviation MSD			Mean Absolute Deviation MAD			Mean Deviation MD		
	Normal	S.S.	A.S.	Normal	S.S.	A.S.	Normal	S.S.	A.S
0.25	0.0658	0.0073	0.0121	0.2277	0.0761	0.1002	-0.2277	-0.0761	-0.1002
0.10	0.0927	0.0023	0.0023	0.2705	0.0426	0.0421	-0.2705	-0.0426	-0.0421
0.075	0.0863	0.0018	0.0011	0.2462	0.0356	0.0267	-0.2462	-0.0318	-0.0198
0.05	0.0718	0.0034	0.0017	0.1946	0.0397	0.0342	-0.1946	-0.0303	0.0012
0.025	0.0424	0.0153	0.0071	0.1712	0.0881	0.0664	0.0065	-0.0635	0.0175
0.01	0.2127	0.2475	0.0917	0.4319	0.4340	0.2356	0.4319	-0.4340	-0.2303
Average	0.0953	0.0463	0.0193	0.2570	0.1194	0.0842	-0.0834	-0.1131	-0.0623

The entries represent summary measures defined by Equations (15), (16) and (17). The smaller the values, the better the estimated mean–VaR frontiers approximate the empirical ones.

Table 4: Portfolio Weights and Realized VaR Levels for Different VaR Limits

	$\text{VaR}_{.01}$ Target Level											
	2			3			4			5		
Asset	Normal	S.S.	A.S.	Normal	S.S.	A.S.	Normal	S.S.	A.S.	Normal	S.S.	A.S.
ALZ	0	NA	NA	0	0	0	0	0	0	0	0	0
BAB	0	NA	NA	0	0	0	0	0	0	0	0	0
BAS	0	NA	NA	0.0448	0.0524	0.0520	0	0.0548	0.0546	0	0.0072	0.0079
BAY	0	NA	NA	0	0.0231	0.0228	0	0.0064	0.0068	0	0	0
BMW	0	NA	NA	0.1531	0.0554	0.0542	0.1400	0.1160	0.1143	0.0100	0.1058	0.1050
COM	0.0099	NA	NA	0	0.0364	0.0367	0	0.0015	0.0019	0	0	0
CON	0.0204	NA	NA	0	0.0027	0.0030	0	0	0	0	0	0
DGX	0.0332	NA	NA	0	0.0316	0.0316	0	0.0079	0.0084	0	0	0
DBK	0	NA	NA	0	0	0	0	0	0	0	0	0
DRD	0.0082	NA	NA	0	0.0237	0.0245	0	0	0	0	0	0
HEN	0.0717	NA	NA	0	0.0240	0.0250	0	0	0	0	0	0
KAR	0.0832	NA	NA	0	0	0	0	0	0	0	0	0
LIN	0.0707	NA	NA	0	0	0	0	0	0	0	0	0
MAN	0	NA	NA	0	0	0	0	0	0	0	0	0
MMN	0.0059	NA	NA	0.1532	0.0637	0.0627	0.1283	0.1040	0.1038	0.0041	0.0916	0.0915
MTL	0	NA	NA	0	0	0	0	0	0	0	0	0
MUE	0.0802	NA	NA	0.0394	0.0763	0.0760	0	0.0481	0.0486	0	0.0020	0.0026
PRS	0.0953	NA	NA	0	0.0401	0.0413	0	0	0	0	0	0
RWE	0.0629	NA	NA	0	0.0291	0.0304	0	0	0	0	0	0
SAP	0.0430	NA	NA	0.4497	0.1881	0.1851	0.7317	0.4889	0.4876	0.9859	0.7425	0.7414
SCH	0.1168	NA	NA	0	0.0658	0.0667	0	0	0	0	0	0
SIE	0	NA	NA	0	0	0	0	0	0	0	0	0
THY	0	NA	NA	0	0	0	0	0	0	0	0	0
VEB	0.1330	NA	NA	0.1598	0.1739	0.1740	0	0.1522	0.1524	0	0.0509	0.0516
VIA	0.1656	NA	NA	0	0.1136	0.1141	0	0.0200	0.0216	0	0	0
VSW	0	NA	NA	0	0	0	0	0	0	0	0	0
Realized VaR	2.55	NA	NA	3.21	2.80	2.78	4.49	3.26	3.25	5.13	4.47	4.47

Column triplets labeled $i = 2, 3, 4, 5$ show the portfolio weights obtained by solving the optimization problem (14) with the $\text{VaR}_{.01}$ constraint set to i. The realized VaR is given by the empirical 1% quantile of the portfolio returns when using the corresponding weights shown in this table.

Table 5:
Backtesting results according to the Basle Committee's three–zone approach over seven non–overlapping 250-day periods

	VaR$_{.01}$ Limit											
	2			3			4			5		
Zone	Normal	S.S.	A.S.	Normal	S.S.	A.S.	Normal	S.S.	A.S.	Normal	S.S.	A.S.
Green	2	NA	NA	4	6	6	5	7	7	4	6	6
Yellow	4	NA	NA	3	1	1	2	0	0	3	1	1
Red	1	NA	NA	0	0	0	0	0	0	0	0	0
Avg. number of violations	5.00	NA	NA	3.71	1.86	1.86	2.86	1.71	1.71	3.57	1.86	1.86

The Green Zone indicates $0-4$ violations of the VaR limit during a 250–day period; the Yellow Zone indicates $5-9$ violations; and the Red Zone means that there are more than 9 violations of the VaR limit.

Figure 1: Mean–VaR Frontiers for Different Target Probabilities

Empirical: · · ·   Normal: – · –   Symmetric Stable: – – –   Asymmetric Stable: —

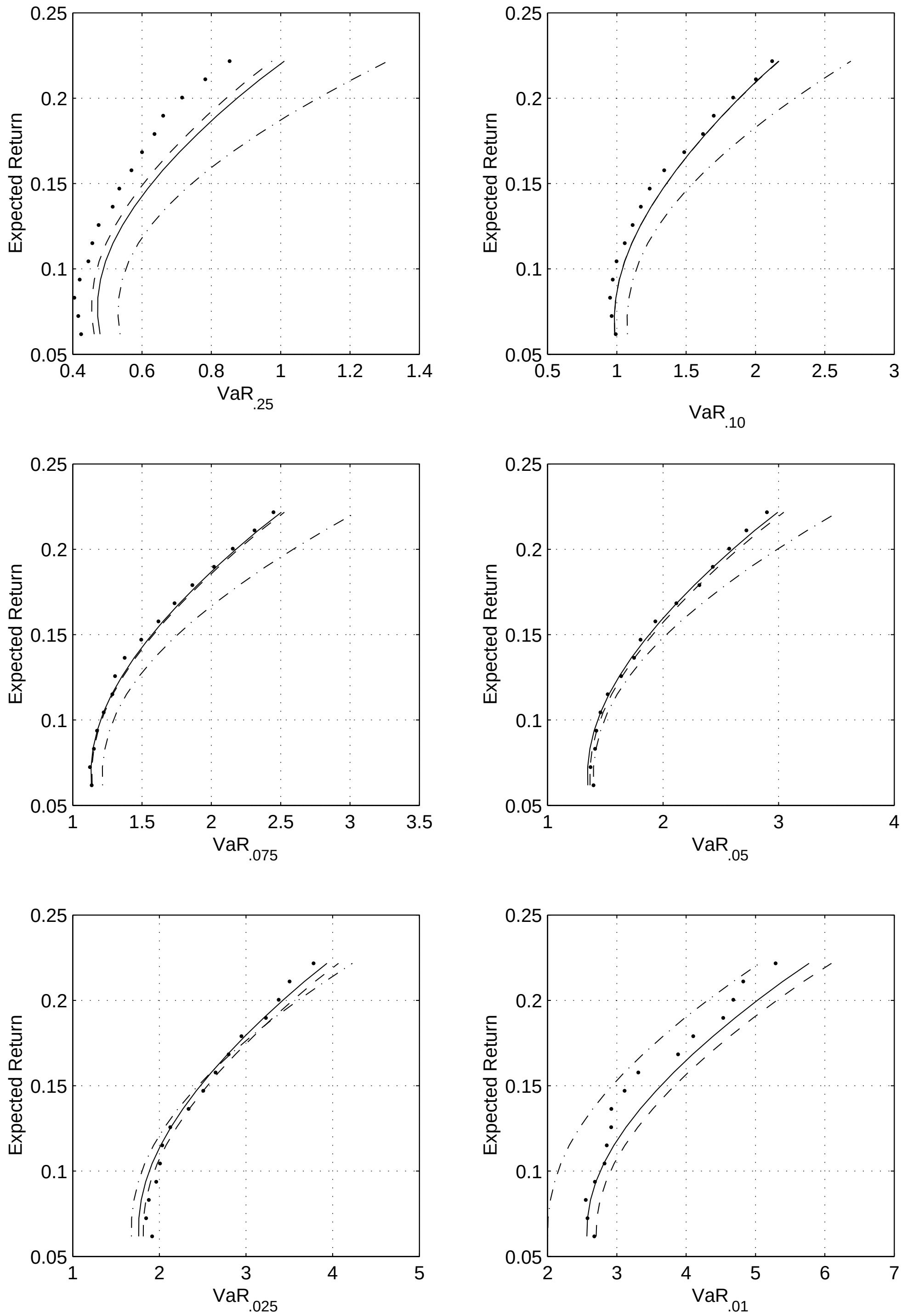

Figure 2: Number of Violations of the $\mathrm{VaR}_{.01}$ Limit During the Preceding 250 Days